Example 10-19: Consider the case where the system function is

$$H(z) = \frac{1 - z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}} \tag{10.18}$$

In order to find the magnitude of the frequency response, we work with the magnitude squared and then take the positive square root at the end. The magnitude squared is derived by multiplying out all the terms in the numerator and denominator of $H(e^{j\hat{\omega}})H^*(e^{j\hat{\omega}})$, and then collecting terms where the inverse Euler formula applies.

$$\begin{aligned}
|H(e^{j\hat{\omega}})|^2 &= H(e^{j\hat{\omega}})H^*(e^{j\hat{\omega}}) \\
&= \frac{1 - e^{-j2\hat{\omega}}}{1 - 0.9e^{-j\hat{\omega}} + 0.81e^{-j2\hat{\omega}}} \cdot \frac{1 - e^{j2\hat{\omega}}}{1 - 0.9e^{j\hat{\omega}} + 0.81e^{j2\hat{\omega}}} \\
&= \frac{2 - 2\cos(2\hat{\omega})}{2.4661 - 3.258\cos\hat{\omega} + 1.62\cos(2\hat{\omega})}
\end{aligned}$$

This formula is useful for computations because it is expressed completely in terms of cosine functions. The procedure is general, so a similar formula could be derived for any IIR filter. Since the cosine is an even function, $|H(e^{j\hat{\omega}})|^2$ is an even function of $\hat{\omega}$, that is,

$$|H(e^{-j\hat{\omega}})|^2 = |H(e^{j\hat{\omega}})|^2$$

This is true for any system function when the difference equation has real coefficients. The magnitude $|H(e^{j\hat{\omega}})|$, which requires a square root, is also an even function.

The phase response is a bit messier. Recall that the angle of a complex number z can be found via $\tan^{-1}\left(\Im\{z\}/\Re\{z\}\right)$. If arctangents are used to extract the angles of the numerator and denominator, then the two angles must be subtracted to get the phase. The filter coefficients in this example are real, so the phase is

$$\angle H(e^{j\hat{\omega}}) = \tan^{-1}\left(\frac{\sin(2\hat{\omega})}{1 - \cos(2\hat{\omega})}\right) - \tan^{-1}\left(\frac{0.9\sin\hat{\omega} - 0.81\sin(2\hat{\omega})}{1 - 0.9\cos\hat{\omega} + 0.81\cos(2\hat{\omega})}\right)$$

which can be shown to be an odd function of $\hat{\omega}$ (i.e., $\angle H(e^{-j\hat{\omega}}) = -\angle H(e^{j\hat{\omega}})$).

The even magnitude and odd phase are general characteristics of the frequency response (or any DTFT) of a real impulse response (or any real sequence). As discussed in Section **??** for the FIR case and now demonstrated for an IIR filter, the frequency response has conjugate symmetry such that

$$H(e^{-j\hat{\omega}}) = H^*(e^{j\hat{\omega}})$$

whenever the impulse response is real.

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