Example 7-6: Consider an ideal LPF with frequency response given by (**??**) and impulse response in (**??**). Now suppose that the input signal $x[n]$ to the ideal LPF is a bandlimited sinc signal

$$x[n] = \frac{\sin \hat{\omega}_b n}{\pi n} \tag{7.1}$$

Working in the time domain, the corresponding output of the ideal LPF would be given by the convolution expression

$$y[n] = x[n] * h_{\text{LP}}[n] \quad = \sum_{m=-\infty}^{\infty} \left(\frac{\sin \hat{\omega}_b m}{\pi m} \right) \left(\frac{\sin \hat{\omega}_{\text{co}}(n-m)}{\pi (n-m)} \right) \tag{7.2}$$

Evaluating this convolution directly in the time domain is impossible both analytically and via numerical computation. However, it is straightforward to obtain the filter output if we use the DTFT because in the frequency domain the transforms are both rectangles and they are multiplied. From (**??**), the DTFT of the input is

$$X(e^{j\hat{\omega}}) = \begin{cases} 1 & |\hat{\omega}| \leq \hat{\omega}_b \\ 0 & \hat{\omega}_b < |\hat{\omega}| \leq \pi \end{cases} \tag{7.3}$$

Therefore, the DTFT of the ideal filter's output is $Y(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}})H(e^{j\hat{\omega}})$, which would be of the form

$$Y(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}})H_{\text{LP}}(e^{j\hat{\omega}}) = \begin{cases} 1 & |\hat{\omega}| \leq \hat{\omega}_a \\ 0 & \hat{\omega}_a < |\hat{\omega}| \leq \pi \end{cases} \tag{7.4a}$$

When multiplying the DTFTs to get the right-hand side of (7.4a), the product of the two rectangles is another rectangle whose width is the smaller of $\hat{\omega}_b$ and $\hat{\omega}_{\text{co}}$, so the bandlimit frequency $\hat{\omega}_a$ is

$$\hat{\omega}_a = \min(\hat{\omega}_b, \hat{\omega}_{\text{co}}) \tag{7.4b}$$

Since we want to determine the output signal $y[n]$, we must take the inverse DTFT of $Y(e^{j\hat{\omega}})$. Thus, using (**??**) to do the inverse transformation, the convolution in (7.2) must evaluate to another sinc signal

$$y[n] = \sum_{m=-\infty}^{\infty} \left(\frac{\sin \hat{\omega}_b m}{\pi m} \right) \left(\frac{\sin \hat{\omega}_{\text{co}}(n-m)}{\pi (n-m)} \right) = \frac{\sin \hat{\omega}_a n}{\pi n} \tag{7.5}$$

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