Lab S-3: Beamforming with Phasors

Pre-Lab: Read the Pre-Lab and do all the exercises in the Pre-Lab section ***prior to attending lab.***

Verification: The Exercise section of each lab should be completed **during your assigned Lab time** and the steps marked *Instructor Verification* signed off **during the lab time**. One of the laboratory instructors must verify the appropriate steps by signing on the **Instructor Verification** line. When you have completed a step that requires verification, simply raise your hand and demonstrate the step to the instructor. Turn in the completed verification sheet before you leave the lab.

Lab Homework Questions: The Lab-Homework Sheet has a few lab related questions that can be answered at your own pace. The completed Lab-HW sheet should be turned in at the beginning of the next lab.

1 Pre-Lab

Objective: Learn how the outputs from an array of two (or more) spatially separated sensors that receive a combined signal from ***multiple sources*** can be used to estimate the directions to the sources. The key processing is ***beamforming*** where the sensor outputs are combined linearly with different phases.

$$\text{Beamforming Output Signal:}\quad b(t) = \sum_{k=1}^{N} \alpha_k r_k(t - \tau_k) \tag{1}$$

The signal received at the k^{th} receiver is $r_k(t)$, α_k is the amplitude weighting, and τ_k is the time shift applied to $r_k(t)$. When used as a beamformer, the array weights and time shifts can be chosen to either eliminate some signals, or to be "steered" in the direction of important signals.

1.1 Overview

There are several specific cases that are considered in this lab:

1. Array response to a single sinusoidal test source that is moved by changing its direction.
2. Array design (weights and time shifts) to steer in a given direction.
3. *Nulling:* When there are several sinusoidal sources at different directions, the receiving array can keep one and remove the others. For example, we can design the time shifts (phases) of the receiving array sensors to null out two (or more) directions by solving simultaneous sinusoidal equations.

1.2 Review of Direction of Arrival (DOA) Sensing

Consider a simple measurement system that consists of two receivers that can both acquire the same source signal. If the sensors are placed a small distance apart, then the signals travel slightly different paths from the source to the receivers. When the travel paths are different distances, the signals arrive at different times. Since time delay corresponds to phase, we say that the received signals arrive "out of phase."

The received signal at one sensor, called $r(t)$, is a delayed copy of the transmitter signal $s(t)$. If the time delay from source to receiver is τ_{sr}, then we can write

$$r(t) = s(t - \tau_{\text{sr}})$$

McClellan, Schafer, and Yoder, *DSP First, 2e*, ISBN 0-13-065562-7.
Prentice Hall, Upper Saddle River, NJ 07458. ©2015 Pearson Education, Inc.

McClellan, Schafer and Yoder, *Signal Processing First*.
Prentice Hall, Upper Saddle River, New Jersey, 2003. ©2003 Prentice Hall.

where $s(\cdot)$ is the transmitted (sinusoidal) signal.[1] The travel time τ_{sr} can be computed easily if we are given the speed of sound (or light) and the locations of the source and receiver(s).

The relative time delay between two receivers can be related to the direction of a source. Consider the case of one source transmitting the signal $s(t)$ to two (or more) receivers. In Fig. 1 the received signals are labelled with subscripts 1, 2 and 3:

$$\begin{aligned} \text{Receiver \#1:} \quad & r_1(t) = s(t - \tau_{sr_1}) \\ \text{Receiver \#2:} \quad & r_2(t) = s(t - \tau_{sr_2}) \\ \text{Receiver \#3:} \quad & r_3(t) = s(t - \tau_{sr_3}) \end{aligned}$$

where τ_{sr_1} is the propagation time from the source to Receiver #1, τ_{sr_2} the propagation time to Receiver #2, and so on.

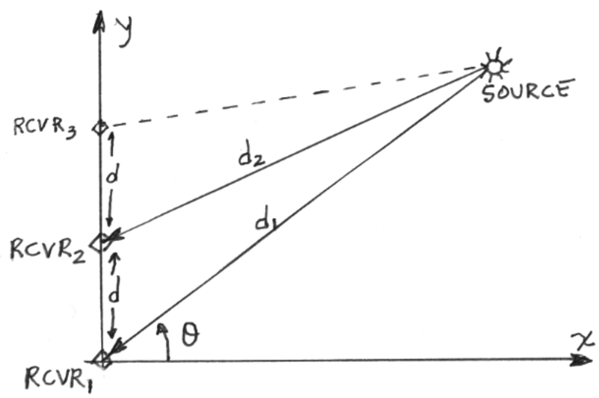

Figure 1: Source with multiple receivers having uniform inter-sensor spacing of d. When $\theta > 0$ the signal arrives first at receiver #3, then #2, and finally #1, i.e., $d_3 < d_2 < d_1$, and the source lies in the first quadrant. When the source is in the fourth quadrant, $\theta < 0$. With receivers on the y-axis it is not possible to distinguish left from right, so we assume that the source never lies in the second or third quadrant.

Suppose that the problem scenario is planar as shown in Fig. 1. The source-to-receiver distances are d_1 for receiver #1, d_2 for receiver #2, and d_3 for receiver #3. When the source is very far away, i.e., $d_i \gg d$, it is possible to derive a very simple *approximate* formula relating d, θ and $(d_i - d_{i+1})$, which is

$$d_1 - d_2 \approx d \sin\theta \tag{2}$$

This difference in propagation distance $(d_i - d_{i+1})$ leads to two observations:

1. The time it takes the source signal to propagate to the two receivers is different. For example, if $d_1 > d_2$ the signal arrives first at receiver #2. Assuming that the velocity of propagation is c, the time difference of arrival (TDOA) is

$$\Delta\tau = \tau_1 - \tau_2 = (1/c)(d \sin\theta) \tag{3}$$

2. When the source is a sinusoidal signal, we can exploit the fact that time delay is related to the phase of the sinusoid, which means that the phase is different at the two receivers. Using the relation $\varphi = -\omega\tau$, it is possible to write a simple formula for the phase difference:

$$\Delta\varphi = \varphi_1 - \varphi_2 = -(\omega/c)(d \sin\theta) \tag{4}$$

[1]For simplicity we ignore propagation losses. Usually, the amplitude of an acoustic signal that propagates over a distance R is reduced by an amount that is inversely proportional to R or R^2.

1.3 Angle-Sensitive Processing

In the Direction of Arrival lab, you can study a two-receiver system that is able to determine direction by comparing phases (4) or time delays (3) at the two sensors. However, this two-receiver method only works when there is a single source signal. A more general type of system that can perform angle-sensitive processing is called a *beamformer*. The attractive feature of a beamformer is that it can provide useful direction estimates even when there are multiple source signals from different directions.

1.3.1 Expectation for Angle-Sensitive Processing

Consider the following three types of angle-sensitive processing:

1. *Finding Multiple Directions:* An ideal Direction Finder (DF) would give an output listing all the source directions that are generating signals that can be sensed by the array.

2. *Scanning All Directions:* Sometimes the sensor array has to test all (or many) directions and make a YES/NO decision as to whether a source signal is present (or not) in each one.

3. *Nulling:* If an adversary is *jamming* the receivers, the beamformer could be configured to eliminate that source. In order to eliminate signals arriving from certain directions, the sensor outputs would be combined with different time shifts to cancel out when summed (i.e., have zero amplitude).

1.3.2 Beamforming

A beamformer works by combining the outputs from several sensors into a single output signal. Since we know that signals received from a source at angle θ are time-delayed with respect to each other, an angle-sensitive beamformer would involve time shifting. Suppose we have signals received at N sensors. The general beamformer does a time shift and weighting of the N received signals prior to addition:

$$b(t, \beta) = \sum_{k=1}^{N} \alpha_k r_k(t - \tau_k) \tag{5}$$

where $r_k(t)$ is the signal received at the k^{th} receiver, α_k is the amplitude weighting, and τ_k is the time shift applied to the output of the k^{th} receiver.

The most common strategy for choosing the time shift is to *align* the signals coming from a source *presumed to be at angle* β. In fact, we could write $\tau_k(\beta)$ to indicate the dependence of the time shift on a presumed source angle. In this strategy, we call β the *beam steering angle* because it can be viewed as the direction of a beam, which is the strongest response of the beamformer (5) when plotted versus angle.

If we concentrate on the two-receiver case, we can work out the concept of signal alignment using our knowledge that the time shift between two receivers separated by d meters is $\Delta\tau = (d/c)\sin\theta$ for a (presumed) source at angle θ. Referring to Fig. 1 if the signal at RCVR #1 is $r_1(t) = s(t - \tau_d)$, then the signal at RCVR #2 would be written as

$$r_2(t) = s(t - \tau_d + (d/c)\sin\theta) \tag{6}$$

where the sign of the $(d/c)\sin\theta$ term is positive because the signal arrives first at RCVR #2 when θ is positive, i.e., when the source lies in the first quadrant. *Note:* τ_d is unknown because we don't know the distance to the source; nevertheless, only the relative time shift between receivers matters.

When steered to the angle β, the primary function of the two-receiver beamformer is to time shift the second signal so that it will be "in phase" with the first signal. One of the signals must be chosen as the reference signal, so we pick RCVR #1 as the reference and apply a relative time shift to the ouptut of RCVR #2 equal to the negative of the time shift for an arriving signal from the *presumed direction* β.

$$b(t, \beta) = \alpha_1 r_1(t) + \alpha_2 r_2(t - (d/c)\sin\beta) \tag{7a}$$

$$= \alpha_1 s(t - \tau_d) + \alpha_2 s(t - \tau_d + \underbrace{(d/c)\sin\theta - (d/c)\sin\beta)}_{\text{time-aligned if } \theta = \beta} \tag{7b}$$

Note the sign difference for the term $\tau_2(\beta) = (d/c)\sin\beta$ in eq. (7a). Also, when the beamformer adds the signals in (7a), no time shift is applied to the reference receiver, i.e., $\tau_1(\beta) = 0$.

We can extend the beamformer idea to a three-receiver array. A very common configuration is a uniformly spaced linear array, where the third receiver would be located at $(0, 2d)$. *The time shift must be relative to the reference receiver* (RCVR #1), so the three-receiver beamformer output is

$$b(t, \beta) = \alpha_1 r_1(t) + \alpha_2 r_2(t - (d/c)\sin\beta) + \alpha_3 r_3(t - 2(d/c)\sin\beta) \tag{8}$$

where the time shifts are $\tau_1(\beta) = 0$, $\tau_2(\beta) = (d/c)\sin\beta$, and $\tau_3(\beta) = 2(d/c)\sin\beta$.

The beamformer processing depends on the angle β which is also called the *steering angle* because we can think of the beamformer as trying to time-align signals coming from the direction β. If β is a true source direction, we get a maximal output because when we add N sinusoids the largest possible amplitude is obtained when the sinusoids are in phase.

1.4 Beamformer Response vs. Source Direction

One way to characterize the beamformer is to measure its response to a source at any possible angle. The following test repeatedly calculates the beamformer output as the direction of a single source is moved:

```
Pick a steering angle (beta in radians)
for sourceDirRadians = -1.57 to 1.57 radians
   synthesize the source signal, s(t), usually a sinusoid
   delay s(t) for each receiver, depending on source-receiver distance
   do the Beamform sum using time shifts from eq.(7)
   record the max amplitude of the Beamform output
end
plot the max amplitude Beamform output versus sourceDirRadians
```

The plot of the maximum amplitude of the beamformer output versus angle (e.g., Fig. 2) is called the *beamformer response*. It shows how the beamformer prefers the steered direction β over other directions. It might also show that there are even some directions where the beamformer output is zero. However, this method of getting the response vs. direction involves a lot of unnecessary work because time signals have to be generated. *There is a major simplification when we use* ***Phasors*** *to represent sinusoids.*

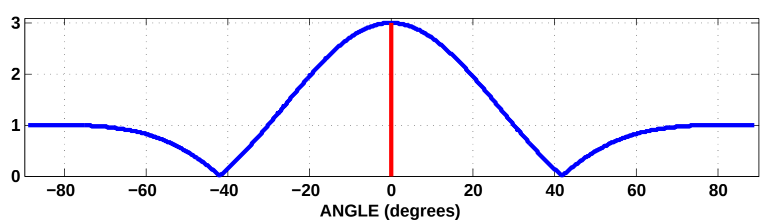

Figure 2: Beamformer response when all three complex weights are equal to $1e^{j0}$. This is the $\beta = 0$ case.

1.5 Beamformer Response Computed with Phasors

A very common situation where beamforming is used is the case of *narrowband beamforming* where the source signal is, in effect, a sinusoid with a known frequency, ω_0. The beamformer, as in (8), adds sinusoids which can be done by adding their phasor representations.

Key Ideas:
- Since the beamformer involves time-shifted (and weighted) receiver signals, in phasor notation these operations become multiplications by a complex number.
- For each presumed source direction, summing the phasors gives the phasor representation of the output sinusoid of the beamformer
- The maximum amplitude of a sinusoid is the magnitude of its phasor representation.

When the source signal (from direction θ) is a sinusoid, the two-receiver beamformer becomes

$$b(t,\beta) = \alpha_1 \cos(\omega_0(t - \tau_1(\beta)) + \psi_{s1}(\theta)) + \alpha_2 \cos(\omega_0(t - \tau_2(\beta)) + \psi_{s2}(\theta)) \tag{9}$$

where ψ_{s1} and ψ_{s2} are the phases of the source signal at the receivers. If there is an actual source at direction θ, these phases depend on the source-receiver distance. However, only the phase difference matters, and eq. (4) shows that the phase difference has a simple dependence on θ.

Now we convert (9) to phasors, and write the beamformer output using the *phasor representation* of the two sinusoids:

$$B(\beta,\theta) = \underbrace{\alpha_1 e^{-j\omega_0\tau_1(\beta)}}_{W_1} e^{j\psi_{s1}(\theta)} + \underbrace{\alpha_2 e^{-j\omega_0\tau_2(\beta)}}_{W_2} e^{j\psi_{s2}(\theta)} \tag{10}$$

Notice that the complex amplitudes in (10) *do not depend on the time variable* t. The terms W_1 and W_2 are complex numbers formed from the weights and time-shifts of the beamformer, so they are called the *complex weights* of the beamformer. This formula (10) for the phasor representation of the beamformer output can be generalized in the obvious way to add more terms if we have more receivers.

Now we return to the problem of computing and plotting the beamformer response versus source direction. We want to plot the maximum amplitude of the beamformer output sinusoid versus source direction. From the complex amplitude (phasor), the maximum amplitude is just the magnitude of the phasor. Why? Thus we can perform one more simplification:

$$|B(\beta,\theta)| = \left|W_1 e^{j\psi_{s1}} + W_2 e^{j\psi_{s2}}\right| \tag{11}$$

$$= \cancel{\left|e^{j\psi_{s1}}\right|} \left|W_1 + W_2 e^{j(\psi_{s2}-\psi_{s1})}\right| \tag{12}$$

The magnitude of the first term is always one. As a test case, when $W_1 = W_2$ the result should have a maximum at zero like Fig. 2.

With the phasor simplification complete, we can rewrite the pseudo-code for computing and plotting the N-receiver beamformer response versus source direction:

```
create complex weights for the array, depending on steering angle (beta)
for sourceDirRadians = -1.57 to +1.57   (directions of source)
   create phase differences across the array, depending on sourceDirRadians
   make Phasors at all receivers with appropriate exponents
   do a sum like eq. (9), but for N receivers
   take magnitude to get the max amplitude, which is the Beamform response
end
plot Beamform response versus sourceDirRadians
```

1.5.1 MATLAB for Beamformer Response

Here is the shell of a MATLAB function derived from the pseudo code above.

```
Wbeta = ???;  % complex Weights from specified steering angle
theta = -1.57:0.001:1.57;
for m=1:length(theta)
  angleSrc = theta(m);
  eJpsi = ???;  % Phasors at receivers from source at angleSrc
  bfOutPhasor(m) = sum(Wbeta.*eJpsi);
end
bfResponseMaxAmp = ???
plot( theta, bfResponseMaxAmp )
```

1.6 Random Parameters Generated by BFgen

A simple MATLAB function is supplied to generate random parameters for your work in this lab.

```
BFgen('UserID',Part)
```

The input is your user ID along with an integer for different parts of the Lab Exercise. The output displayed in the MATLAB command window gives the values of the parameters needed in one of the specific sections below. The function has been converted to MATLAB's p-code format because the parameter generation process involves knowing the solution to the exercise.

2 Lab Exercise

For the instructor verification, you should demonstrate understanding of concepts in a given subsection by answering questions from your lab instructor (or TA). It is not necessary to do everything in the subsections, i.e., skip parts that you already know. The Instructor Verification is usually placed close to the most important item, i.e., the one most likely to generate questions from the instructors.

2.1 Complex Weights of 3-Receiver Beamformer

In this part, use the `BFgen('UserID',1)` function with its second argument set to 1 to generate a steering angle (β), along with an intersensor distance d, a frequency ω_0, and the propagation velocity c. Use this information to produce three complex weights for a 3-receiver beamformer steered to the given angle β. Assume that the amplitudes α_k in this beamformer are all equal to one. Write your values for the complex weights in polar form on the verification page, along with a brief explanation.

Instructor Verification (separate page)

2.2 3-Receiver Beamformer Response vs. Angle

2.2.1 MATLAB code

Using the complex weights from the previous part, write a `for` loop that calculates the beamformer output (maximum amplitude) for angles $-1.57 \le \theta \le 1.57$ (spaced by 0.001 rads). Write the MATLAB code on your verification sheet (by hand).

Instructor Verification (separate page)

2.2.2 Plot Beamformer Performance

Then make a plot of the beamformer response versus angle using MATLAB, and also sketch the plot by hand on the verification page. Annotate the sketched plot carefully, showing the peak value(s) of the beamformer output and the angle(s) where peak value(s) are found. Also, give values of any angles where the beamformer output is zero,[2] i.e., *nulling directions.*

Instructor Verification (separate page)

2.2.3 Sinusoidal Output of Beamformer

Finally, let the source direction be $\theta = 0$. For beamformer designed in Sect. 2.1, determine a formula for the output signal, which is a sinusoid. Give exact values for its frequency and amplitude, but write the phase as the symbol φ. Explain how you can use the plot in the previous part to answer this question.

Instructor Verification (separate page)

2.3 Design Beamformer with Nulling Constraints

For Lab-HW, consider the following design question. Create the complex weights for a 3-receiver beamformer that satisfies the following specifications:

1. For a source at angle θ_1, or at angle θ_2, the beamformer output should be zero, i.e.,

$$\alpha_1 \cos(\omega_0(t-\tau_1)+\psi_{s1}(\theta_1)) + \alpha_2 \cos(\omega_0(t-\tau_2)+\psi_{s2}(\theta_1)) + \alpha_3 \cos(\omega_0(t-\tau_3)+\psi_{s3}(\theta_1)) = 0$$
$$\alpha_1 \cos(\omega_0(t-\tau_1)+\psi_{s1}(\theta_2)) + \alpha_2 \cos(\omega_0(t-\tau_2)+\psi_{s2}(\theta_2)) + \alpha_3 \cos(\omega_0(t-\tau_3)+\psi_{s3}(\theta_2)) = 0$$

2. For a source at angle θ_0, the beamformer output should be equal to $7\cos(\omega_0 t)$.

3. The three receivers are uniformly spaced at $(0, 0)$, $(0, d)$ and $(0, 2d)$.

4. Assume the propagation velocity (c) and source signal frequency (ω_0) are known.

Write a set of ***simultaneous phasor equations*** that captures all the information in the specifications 1–4. It is only necessary to set up the equations because without numbers no easy solution is possible. The equations should be written in matrix form $\mathbf{Aw} = \mathbf{b}$ where $\mathbf{A}$ is a 3×3 matrix with complex-valued entries, $\mathbf{w}$ is the unknown vector of beamformer (complex) weights, and $\mathbf{b}$ is a known vector.
Comments: It is essential to distinguish the knowns from the unknowns. Equations such as those in step 1 above define how the beamformer processes a signal from direction θ_1 or θ_2. The direction angles $\{\theta_0,\ \theta_1,\ \theta_2\}$ are known because we want the beamformer to have a specified output for those directions. Also, the phases such as $\{\psi_{s1}(\theta_1),\ \psi_{s2}(\theta_1),\ \psi_{s3}(\theta_1)\}$ are known because these are the phases at the receivers from one source signal direction. These phases depend on the distances from the source to the receivers. Even though these distances are not known, we have shown previously that when the source is far away, only the relative phases matter for the beamformer. For example, refer to the discussion leading to (12), where the phasor for the arriving signal was written in terms of phase differences.
The unknowns are the parameters of the beamformer which are the weights $\{\alpha_1,\ \alpha_2,\ \alpha_3\}$ and the time shifts $\{\tau_1,\ \tau_2,\ \tau_3,\}$ for the three receivers of the array. The resulting phasors that depend on these unknowns are called the complex weights of the beamformer.

Note: In this nulling design, these complex weights are **not** obtained using the beam steering strategy.

[2]Since the beamformer response vs. direction plot uses a discrete grid for direction (along the horizontal axis), there may not be exact zeros. Instead, you must look for extremely small values of the beamformer response plot.

Lab: Beamforming
INSTRUCTOR VERIFICATION SHEET

Turn this page in to your instructor before the end of your scheduled Lab time.

Name: ______________________________ UserID: _______________ Date: _______

Part 2.1 Run the `BFgen('UserID',1)` function to obtain the system parameters:

$\beta =$ $d =$ $c =$ $f =$ Hz

List the Complex Weights for the beamformer in *polar form*. Discuss your method and results.

Verified:______________________ Date/Time:____________

Part 2.2 Write a MATLAB `for` loop to calculate the beamformer output versus direction.

Verified:______________________ Date/Time:____________

Part 2.2 Sketch a plot of beamformer output versus angle. Annotate the sketch carefully, for peaks and nulls.

Verified:______________________ Date/Time:____________

Part 2.2 When the steering angle (β) is $\beta = 0$, write a formula for the output signal from the beamformer.

Verified:______________________ Date/Time:____________

Lab: Beamforming
LAB HOMEWORK QUESTION

Turn this page in to your instructor at the very beginning of your next scheduled Lab time.

Name: ______________________________ UserID: _______________ Date: _______

Create the complex weights for a 3-receiver beamformer that satisfies the specifications given in Part 2.3 The equations should be written in matrix form $\mathbf{A}\mathbf{w} = \mathbf{b}$ where $\mathbf{A}$ is a 3×3 matrix with complex-valued entries, $\mathbf{w}$ is the vector of beamformer (complex) weights, and $\mathbf{b}$ is a vector.
In addition to the final answer, give some key steps in the derivation leading to the answer.