Lab P-15d: Filter Design with PeZ: IIR Bandpass Filters

Pre-Lab: Read the Pre-Lab and do all the exercises in the Pre-Lab section ***prior to attending lab.***

Verification: The Warm-up section of each lab should be completed **during supervised Lab time**. When you have completed a step that requires verification, demonstrate the result to your instructor and answer any questions about it. Turn in the completed verification sheet before you leave the lab.

Lab Report: The project requires a MATLAB programming effort and should be documented with a lab report that includes a cover sheet, commented MATLAB code, explanations of your approach, and conclusions.

1 PeZ: Introduction

The objective for this lab is to build an intuitive understanding of IIR filter design based on the relationship between the location of poles and zeros in the z-domain, the impulse response $h[n]$ in the n-domain, and the frequency response $H(e^{j\hat{\omega}})$ (the $\hat{\omega}$-domain). A graphical user interface (GUI) called **PeZ** was written in MATLAB for doing interactive explorations of the three domains.[1] **PeZ** is based on the system function, represented as a ratio of polynomials in z^{-1}, which can be expressed in either factored or expanded form as:

$$H(z) = \frac{B(z)}{A(z)} = G\,\frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{\ell=1}^{N}(1 - p_\ell z^{-1})} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{\ell=1}^{N} a_\ell z^{-\ell}} \tag{1}$$

where M is the number of zeros and N the number of poles.

1.1 Controls for PeZ using `pezdemo`

The **PeZ** GUI is controlled by the `Pole-Zero Plot` where the user can add (or delete) poles and zeros, as well as move them around with the pointing device. For example, Fig. 1 shows a second-order FIR filter with two zeros, while Fig. 2 shows a case where two (complex-conjugate) poles have been added, along with two zeros on the unit circle. The buttons named `PP` and `ZZ` were used to add these poles and zeros. By default, the `Add with Conjugate` property is turned on, so poles and zeros are typically added in pairs to satisfy the complex-conjugate property:

A polynomial with real coefficients has roots that are real, or occur in complex-conjugate pairs.

To learn about the other controls in `pezdemo`, access the menu item called "Help" for extensive information about all the **PeZ** controls and menus. Here are a few things to try. You can use the `Pole-Zero Plot` to selectively place poles and zeros in the z-plane, and then observe (in the other plots) how their placement affects the impulse and frequency responses. In **PeZ** an individual pole/zero pair can be moved around and the corresponding $H(e^{j\hat{\omega}})$ and $h[n]$ plots will be updated as you drag the pole (or zero). The red ray in the z-domain window is tied to the red vertical lines on the frequency responses, and they move together. This helps identify frequency domain features that are caused by pole locations or zero locations, because the

[1] **PeZ** was written originally by Craig Ulmer, and is now maintained by Greg Krudysz.

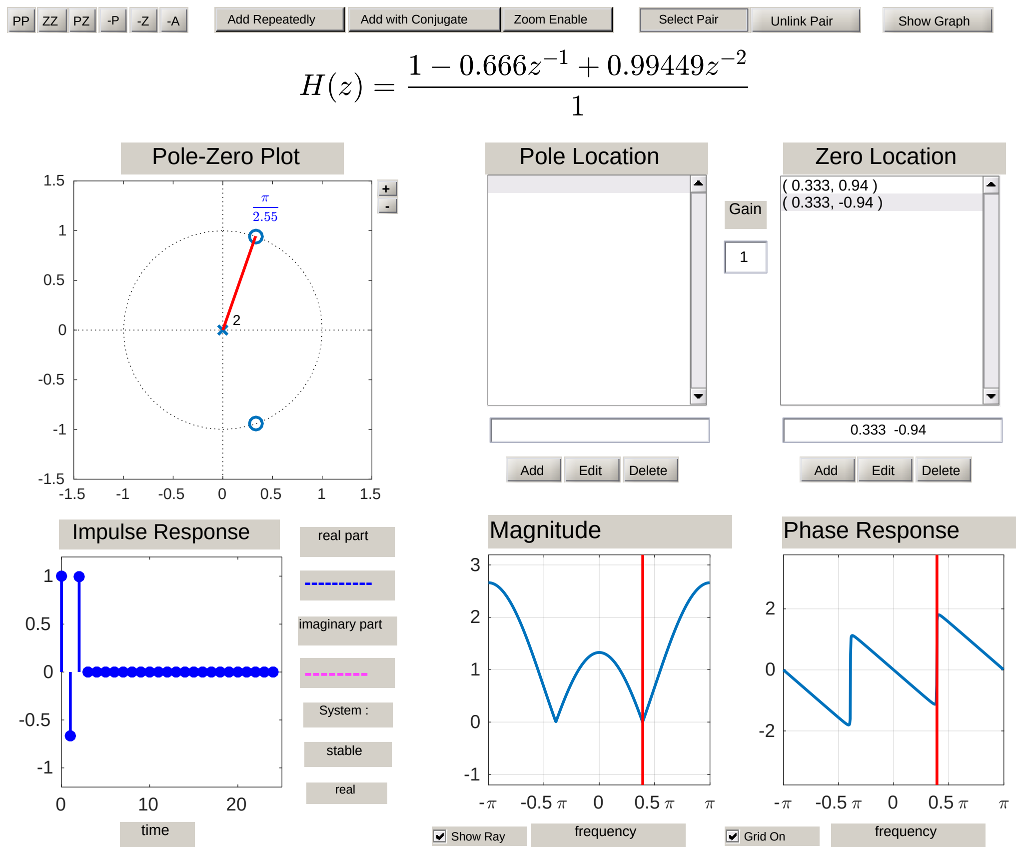

Figure 1: GUI interface for `pezdemo` (ver 2.9) running in MATLAB 2016a. A length-3 FIR filter is shown. Zero locations are given in rectangular coordinates. The angle of the red ray in the pole-zero plot is equal to the frequency $\hat{\omega}$.

angle around the unit circle corresponds to frequency $\hat{\omega}$. Since exact placement of poles and zeros with the mouse is difficult, an **`Edit`** button is provided for numerical entry of the real and imaginary parts. Before you can edit a pole or zero, however, you must first select it in the list of **`Pole Locations`** or **`Zero Locations`**. Removal of individual poles or zeros can also be performed by using the **`-P`** or **`-Z`** buttons, or with the **`Delete`** button. Note that all poles and/or zeros can be easily cleared by clicking on the **`-A`** button.

2 Pre-Lab

To gain some familiarity with the **PeZ** interface, here are two suggested filters that you can create.

2.1 Create an FIR Filter with PeZ

Implement the following FIR system:

$$H(z) = 1 - z^{-1} + z^{-2}$$

by factoring the polynomial and placing the two zeros correctly. Observe the following two facts:

- The impulse response $h[n]$ values (for FIR) are equal to the polynomial coefficients of $H(z)$.

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McClellan, Schafer and Yoder, *Signal Processing First*.
Prentice Hall, Upper Saddle River, New Jersey, 2003. ©2003 Prentice Hall.

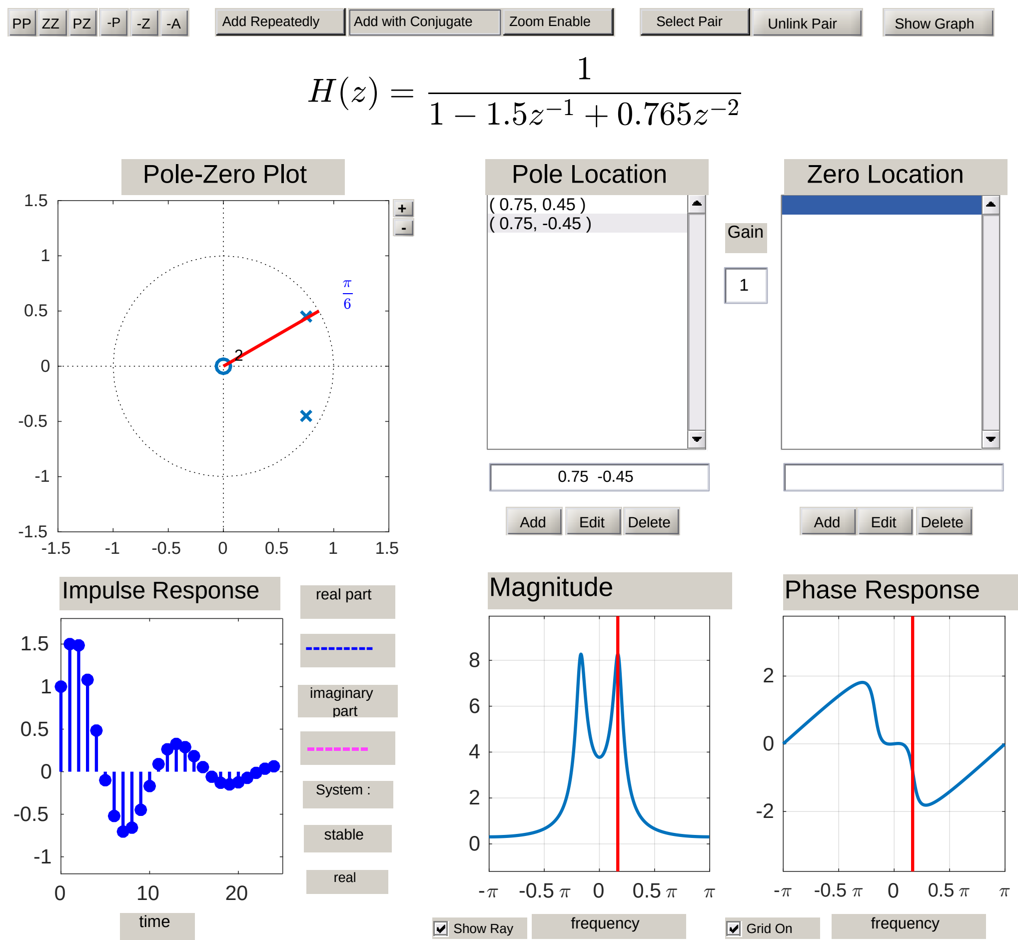

Figure 2: GUI interface for `pezdemo` showing a second-order IIR filter. Pole and zero locations are given in rectangular coordinates. The "Gain" could be adjusted to make the peak of the frequency response equal to one in the passbands.

- The frequency response has nulls because the zeros of $H(z)$ lie exactly on the unit circle. Compare the frequencies of the nulls to the angles of the zeros

Move the zero-pair around the unit circle and observe that the location of the null also moves.

2.2 Create an IIR Filter with PeZ

Implement the following first-order IIR system:

$$H(z) = \frac{1 - z^{-1}}{1 + 0.9z^{-1}}$$

by placing its pole and zero at the correct locations in the z-plane. First try placing the pole and zero with the mouse, and then use the **`Edit`** feature to get exact locations. Since **PeZ** wants to add complex-conjugate pairs, you might have to delete one of the poles/zeros that were added; or you can turn off the **`Add with Conjugate`** feature. Look at the frequency response and determine what kind of filter you have.

Now, use the mouse to "grab" the pole and move it from $z = -0.9$ to $z = +0.8$. To move along the real axis, you can use `Options -> Move on Real Line` from the GUI menu. Observe how the frequency

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response changes. Describe the type of filter that you have created (i.e., HPF, LPF, or BPF).

2.3 Create a Second-Order IIR Filter with PeZ

Use the **PeZ** interface to implement the following second-order system:

$$H(z) = \frac{1 - z^{-2}}{1 + 0.8z^{-1} + 0.64z^{-2}}$$

by determining where the two poles and two zeros are located and then placing the poles and zeros at the correct locations in the z-plane. First try placing the poles and zeros with the mouse, and then use the `Edit` feature to get exact locations. Since **PeZ** wants to add complex-conjugate pairs, you should only have to add one of the poles; for the zeros, the `Add with Conjugate` feature should be turned off because you will be adding two real-valued zeros.

Look at the frequency response and determine what kind of filter you have.

2.4 Not Always Bandpass Filters

It is tempting to think that with two poles the frequency response always has a peak, but there are two interesting cases where that doesn't happen: (1) all-pass filters where $|H(e^{j\hat{\omega}})|$ = constant, and (2) IIR notch filters that null out one frequency, but are relatively flat across the rest of the frequency band.

• Implement the following second-order system:

$$H(z) = \frac{64 + 80z^{-1} + 100z^{-2}}{1 + 0.8z^{-1} + 0.64z^{-2}}$$

by determining where the two poles and two zeros are located and then placing the poles and zeros at the correct locations in the z-plane.
⇒ Look at the frequency response and determine what kind of filter you have.[2]

• Now, use the mouse to "grab" the zero-pair and move the zeros to be exactly on the unit-circle at the same angle as the poles. Observe how the frequency response changes. In addition, determine the $H(z)$ for this filter.

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + 0.8z^{-1} + 0.64z^{-2}}$$

⇒ Describe the type of filter that you have now created.

3 Warm-up

In the warm-up, you will use **PeZ** to create filters with complex conjugate poles and zeros. These are called *second-order filters* because the numerator and/or denominator polynomial is a quadratic with two roots.

3.1 Relationships between z, n, and $\hat{\omega}$ domains

Work through the following exercises and keep track of your observations by filling in the worksheet at the end of this assignment. In general, you want to make note of the following observations:

- Is the length of $h[n]$ finite or infinite, i.e., FIR or IIR?

[2]The relationship between the poles and zeros of an all-pass filter is zero = 1/(pole)*; this situation where two poles and two zeros are linked together can be done with the `PZ` option in **PeZ**.

- *Frequency Domain:* How does $H(e^{j\hat{\omega}})$ change with respect to null location, peak location and/or peak width?
- *Time Domain:* How does $h[n]$ change with respect to its rate of decay for IIR filters? For example, when the impulse response is of the form $h[n] = a^n u[n]$, the impulse response will fall off more rapidly when a is smaller; and if $|a| > 1$ the impulse response blows up.
- *Time Domain:* For second-order IIR filters, $h[n]$ will exhibit an oscillating component, what is the period of oscillation? Also, estimate the decay rate of the "envelope" that overlays the oscillation.

Note: review the "Three-Domains - FIR" and "Three-Domains - IIR" under the Demos link for movies and examples of these relationships.

3.2 Multiple Real Poles and Zeros

(a) Use **PeZ** to place poles at $z = 0.9$ and $z = -0.9$; two zeros will be placed automatically at $z = 0$. You may have to use the `Edit` button to get the locations exactly right. Describe the important features of the frequency response magnitude $|H(e^{j\hat{\omega}})|$, e.g., where passbands are located.

(b) Write out the expression for $H(z)$ created in the previous part. *Hint:* use MATLAB's `poly` function.

(c) Write out the difference equation for the IIR filter whose system function is $H(z)$ in the previous part.

(d) Use **PeZ** to place zeros at $z = 1$ and $z = -1$; two poles will be placed automatically at $z = 0$. Describe the important features of the frequency response magnitude $|H(e^{j\hat{\omega}})|$, and state whether the filter is a LPF, HPF and BPF.

Instructor Verification (separate page)

3.3 Complex Poles and Zeros

PeZ assumes real coefficients for the numerator and denominator polynomials when the `Add with Conjugate` mode is on (which it is by default). Therefore, if we enter a complex pole or zero, **PeZ** will automatically insert a second root at the conjugate location, i.e., $z = \frac{1}{3} + j\frac{1}{2}$ would be accompanied by $z = \frac{1}{3} - j\frac{1}{2}$.

(a) State the property of the polynomial coefficients of $A(z) = 1 - a_1 z^{-1} - a_2 z^{-2}$ that will guarantee that the two roots of $A(z)$ are either both real, or are complex conjugates of each other.

(b) Clear all the poles and zeros from **PeZ**, and then place a zero pair at $z = e^{\pm j2\pi/3}$ Note that **PeZ** automatically places a conjugate pole in the z-domain. The frequency response has an exact null—record the frequency (location) of this null.

(c) Grab the zero and move it around the unit circle; notice how the frequency response changes. Suppose that you want a null in the frequency response at $\hat{\omega} = \pm\pi/2$, determine the zero locations.

(d) Clear all the poles and zeros from **PeZ**, and then place a pole pair at $-0.6 \pm j0.6$, and zeros at $z = \pm 1$. Note that **PeZ** automatically places a conjugate pole in the z-domain. Determine the radius and angles of the poles. The frequency response has a peak—record the frequency location ($\hat{\omega}$) of this peak.

(e) Write out the expression for $H(z)$ created in the previous part. *Hint:* use MATLAB's `poly` function.

(f) Write out the difference equation for the IIR filter whose system function is $H(z)$ in the previous part.

(g) Start again with the pole pair at $-0.6 \pm j0.6$, and zeros at $z = \pm 1$. Decrease the radial distance of the poles from the origin (by dragging), e.g., try $-0.5 \pm j0.5$, and then $-0.4 \pm j0.4$. If you use the `Pole Location` edit window to change the values, the two poles will be "unlinked" and you will have to edit them separately. Therefore, dragging is a more informative way to do this even though it's less precise. Describe the changes in both $h[n]$ and $|H(e^{j\hat{\omega}})|$, as you reduce the pole radius.

(h) Increase the radius of the poles by pushing them closer to the unit circle, and then move the poles outside the unit circle. When the pole-pair is outside the unit circle, describe what happens to $h[n]$.

Instructor Verification (separate page)

4 Lab Exercise: Bandpass Filter Design for IIR

It is easy to design a narrow passband IIR filter by putting a complex pole-pair near the unit circle.

4.1 Complex Poles

The first exercise is to move one pole-pair around and obtain formulas for how the frequency response changes as a function of the pole-pair radius and angle.

(a) Place a single pole-pair at $z = 0.9e^{\pm j0.75\pi}$, and zeros at $z = \pm 1$. Then determine the coefficients of the numerator and denominator polynomials in the resulting $H(z)$.

(b) Make a plot of the frequency response (magnitude only) with `freqz` and measure the two passband edges on either side of the peak. These will be called the *upper passband edge*, $\hat{\omega}_{pu}$, and the *lower passband edge*, $\hat{\omega}_{p\ell}$; obviously, $\hat{\omega}_{p\ell} < \hat{\omega}_{pu}$.

This presents a problem because we must define where to measure the edges of the passband. The usual definition is to measure the edges at the "3-dB level." What this means is that the measurement must be made at a level defined with respect to the peak value of the frequency response. If the peak value is $H_{\max}$, then the "3-dB level" is at $0.707H_{\max}$.[3]

(c) *Definition:* The passband is the frequency region $\hat{\omega}_{p\ell} \leq \hat{\omega} \leq \hat{\omega}_{pu}$, so the *passband width* is $\hat{\omega}_{pu} - \hat{\omega}_{p\ell}$. Write a script to view the frequency response as you move the pole-pair so that the angles remain fixed at $\pm 0.75\pi$, but the pole radius moves to $r = 0.95$, $r = 0.975$ and $r = 0.99$. In each case, measure the passband edges and compute the 3-db width of the peak. To get a precise measurement, use a dense frequency grid with `freqz`, and zoom in on the plot near the passband peak.
$\Rightarrow$ Present your measurements and calculations as a table that summarizes all the cases.

(d) It is possible create a simple formula for how the passband width depends on $(1 - r)$, which is the closeness to the unit circle. One formula that works quite well is the following:

$$\text{PassbandPeakWidth} \approx K(1 - r)/\sqrt{r}$$

where K is a constant of proportionality. Using the measured values from the previous part, determine one value for K that seems to be the best for all the measured cases. For example, averaging might give a useful answer, but it's not the only way.

(e) Move the pole-pair (at a fixed radius $r = 0.95$) so that its angles change from $\pm 0.6\pi$ to $\pm 0.5\pi$ and then to $\pm 0.4\pi$. State a formula for the peak location as a function of the changing pole location.

[3]The frequency response is often plotted on a logarithmic scale using decibels, i.e., $20\log_{10}|H(e^{j\hat{\omega}})|$. If you compute $20\log_{10}(1/\sqrt{2})$ you get -3.01 dB, and $1/\sqrt{2} \approx 0.707$.

4.2 Passband and Stopband

We can characterize general bandpass filters if we define the passband width to be equal to the 3-dB width (as in the previous section). We also need a definition for the stopband edges, so we will arbitrarily define the stopbands of this BPF to be those regions where the frequency response (magnitude) is below -20 db, which is equivalent to 10% of the peak value. Recall that $20\log_{10}(0.1) = -20\,\text{dB}$.

(a) Determine the stopband regions for three of the filters designed in the previous section. Use the four cases where the pole angles are $\pm 0.75\pi$ and the radii are $r = 0.9, 0.95, 0.975$ and 0.99. In each case, measure the edges of the frequency regions for the two stopbands. There is one lower stopband for $0 \le \hat{\omega} \le \hat{\omega}_{s_1}$, and one upper stopband for $\hat{\omega}_{s_2} \le \hat{\omega} \le \pi$. Summarize the measurements in a table.

(b) Usually, filter design becomes difficult when we want the passband and stoband edges to be very close to one another. The difference between neighboring passband and stopband edges is called the *Transition Width*, e.g., $\hat{\omega}_{s_2} - \hat{\omega}_{pu}$ for the upper transition zone. Use the previous bandedge measurements for the passband and stopband to calculate the upper and lower transition widths. Summarize the results in a table that lists the two transition widths versus r for each of the four filters.

(c) Derive a simple *approximate* formula for the transition width vs. r. Use your table of transition widths to determine whether the transition widths depend directly, or inversely, on $(1 - r)$ or r. Determine the two formulas which will probably be slightly different for the upper and lower transition zones.

4.2.1 Design a Bandpass Filter Based on Analog Frequencies

One last question that relates to your understanding of sampling as well as digital filtering.

(a) Design a BPF whose passband is $2900 \le f \le 3100\,\text{Hz}$ when the sampling rate is 8000 Hz using the IIR method above. This requires that you determine the pole and zero locations, i.e., determine the value of r, and the angles of the poles. For the numerator, use two zeros: one at DC and the other at $z = -1$. As a reminder, you are designing a digital filter to be used as the system $H(z)$ in Fig. 3.

(b) Once you have the filter $H(e^{j\hat{\omega}})$, determine its stopbands and give the stopband edges in hertz.

(c) Design a second IIR BPF whose passband lies below 2900 Hz and does not overlap with the first IIR BPF. The second IIR BPF should use $r = 0.9$, along with zeros at $z = \pm 1$. Determine two pole locations for the second filter so that its *entire passband lies within the lower stopband* of the first IIR BPF. Finally, convert the passband edges of the second IIR BPF to continuous-time frequency in hertz.

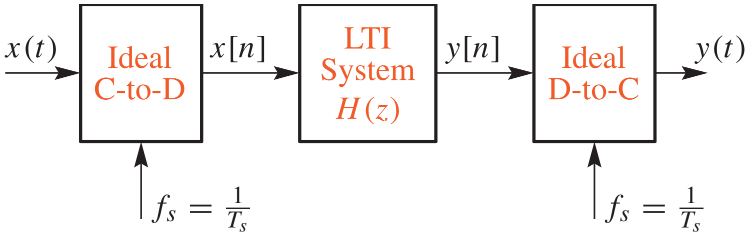

Figure 3: Filtering the analog signal $x(t)$ with a digital filter whose system function is $H(z)$.

4.3 Design BPF with **filterdesign** GUI

Higher order IIR filters can be designed with a software tool such as the **filterdesign** GUI, which has an option for Butterworth filters. The filter specifications are entered in boxes to the right of the figures. A fourth order Butterworth filter that meets the filter specifications given in Section 4.2.1, is shown in Fig. 4. The pole-zero plot is obtained in the **filterdesign** GUI by right-clicking in the plot area.

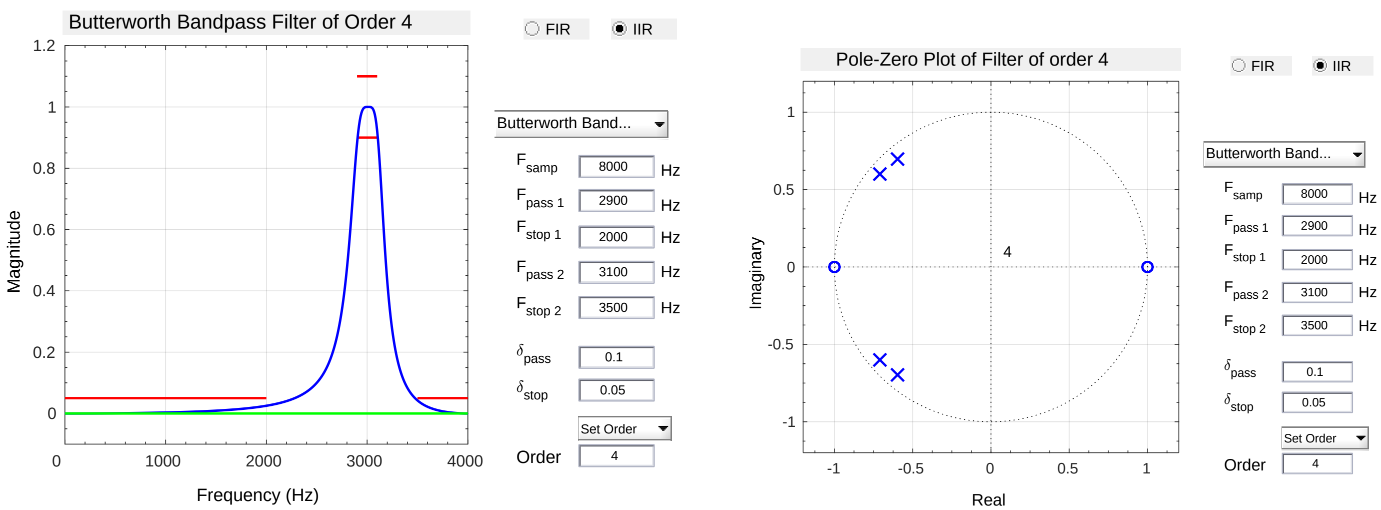

Figure 4: GUI interface for **filterdesign** (ver 2.83) running in MATLAB 2016a. A 4-th order IIR Butterworth filter is shown. (a) Magnitude of the frequency response showing the pass band near 3000 Hz, as well as the two stopbands. The pass band specification is $2900 \leq f \leq 3100$ Hz. The stop bands are $0 \leq f \leq 2000$ Hz and $3500 \leq f \leq 4000$ Hz. (b) Pole-zero plot showing four poles near the unit circle in order to form the pass band. There are four zeros: two at $z = +1$ and two at $z = -1$.

Lab: Filter Design with PeZ: IIR Bandpass Filters
WORKSHEET & VERIFICATION PAGE

For each verification, be prepared to explain your answer and respond to other related questions that the lab TA's or professors might ask. Turn this page in at the end of your lab period.

Name: ______________________________ Date of Lab: __________

Part	Observations from **PeZ**
3.2(a)	Describe Frequency Response; esp. passband location(s):
3.2(b)	$H(z) =$
3.2(c)	Difference Equation:
3.2(d)	Describe Frequency Response:

Verified:____________________ Date/Time:__________

3.3(a)	State Property for Real Roots:
3.3(b)	Location of Null:
3.3(c)	Describe Frequency Response changes, and Zero Locations for new Null:
3.3(d)	Location of Frequency Response Peak:
3.3(e)	$H(z) =$
3.3(f)	Difference Equation:
3.3(g)	Describe changes in $h[n]$ and $\lvert H(e^{j\hat{\omega}})\rvert$
3.3(h)	Poles outside Unit Circle:

Verified:____________________ Date/Time:__________