

Signal Processing First

Lecture 8 Sampling & Aliasing

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1

READING ASSIGNMENTS

- This Lecture:
 - Chap 4, Sections 4-1 and 4-2
 - Replaces Ch 4 in DSP First, pp. 83-94
- Other Reading:
 - Recitation: Strobe Demo (Sect 4-3)
 - Next Lecture: Chap. 4 Sects. 4-4 and 4-5

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3

LECTURE OBJECTIVES

- SAMPLING can cause ALIASING
 - Sampling Theorem
 - Sampling Rate > 2(Highest Frequency)
- Spectrum for digital signals, $x[n]$
 - Normalized Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi \ell$$

 **ALIASING**

SYSTEMS Process Signals



- PROCESSING GOALS:
 - Change $x(t)$ into $y(t)$
 - For example, more BASS
 - Improve $x(t)$, e.g., image deblurring
 - Extract Information from $x(t)$

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5

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4

System IMPLEMENTATION

ANALOG/ELECTRONIC:

- Circuits: resistors, capacitors, op-amps



DIGITAL/MICROPROCESSOR

- Convert $x(t)$ to **numbers** stored in memory



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6

SAMPLING $x(t)$

SAMPLING PROCESS

- Convert $x(t)$ to **numbers** $x[n]$
- “ n ” is an integer; $x[n]$ is a sequence of values
- Think of “ n ” as the storage address in memory

UNIFORM SAMPLING at $t = nT_s$

- IDEAL: $x[n] = x(nT_s)$



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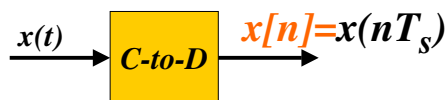
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7

SAMPLING RATE, f_s

SAMPLING RATE (f_s)

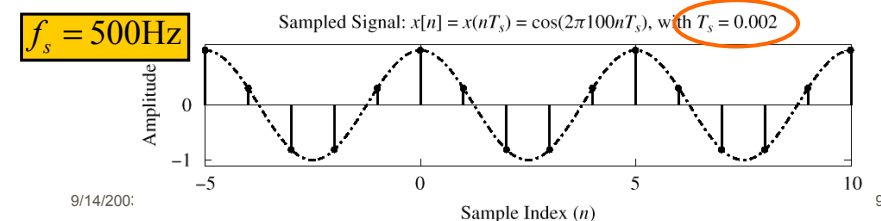
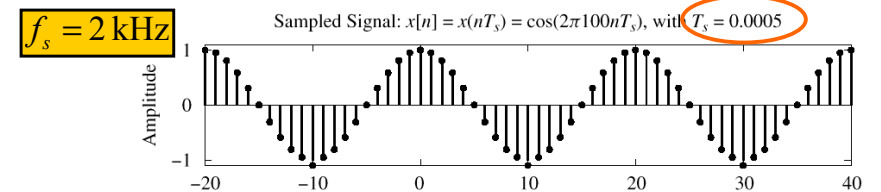
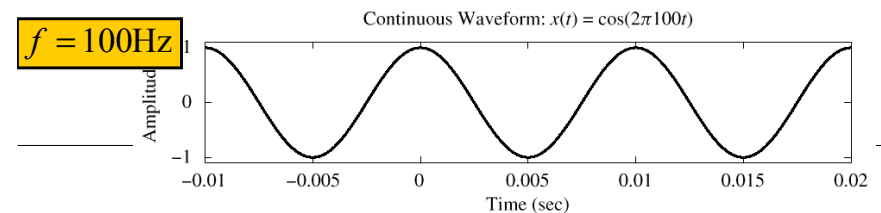
- $f_s = 1/T_s$
 - NUMBER of SAMPLES PER SECOND
- $T_s = 125 \text{ microsec} \rightarrow f_s = 8000 \text{ samples/sec}$
 - UNITS ARE HERTZ: 8000 Hz
- UNIFORM SAMPLING at $t = nT_s = n/f_s$
 - IDEAL: $x[n] = x(nT_s) = x(n/f_s)$



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8



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9

SAMPLING THEOREM

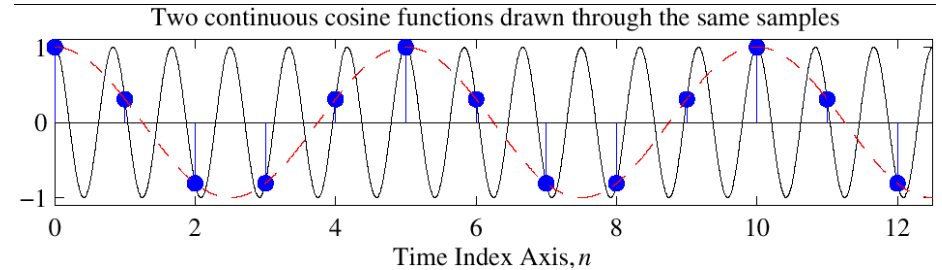
- HOW OFTEN ?
 - DEPENDS on FREQUENCY of SINUSOID
 - ANSWERED by SHANNON/NYQUIST Theorem
 - ALSO DEPENDS on "**RECONSTRUCTION**"

Shannon Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than f_{\max} can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\max}$.

Reconstruction? Which One?

Given the samples, draw a sinusoid through the values



$$x[n] = \cos(0.4\pi n)$$

When n is an integer
 $\cos(0.4\pi n) = \cos(2.4\pi n)$

STORING DIGITAL SOUND

- $x[n]$ is a SAMPLED SINUSOID
 - A list of numbers stored in memory
- EXAMPLE: audio CD
- CD rate is 44,100 samples per second
 - 16-bit samples
 - Stereo uses 2 channels
- Number of bytes for 1 minute is
 - $2 \times (16/8) \times 60 \times 44100 = 10.584$ Mbytes

DISCRETE-TIME SINUSOID

- Change $x(t)$ into $x[n]$ **DERIVATION**

$$x(t) = A \cos(\omega t + \varphi)$$

$$x[n] = x(nT_s) = A \cos(\omega n T_s + \varphi)$$

$$x[n] = A \cos((\omega T_s)n + \varphi)$$

$$x[n] = A \cos(\hat{\omega} n + \varphi)$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s} \quad \text{DEFINE DIGITAL FREQUENCY}$$

DIGITAL FREQUENCY $\hat{\omega}$

- $\hat{\omega}$ VARIES from 0 to 2π , as f varies from 0 to the sampling frequency
- UNITS are radians, not rad/sec
 - DIGITAL FREQUENCY is NORMALIZED

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}$$

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14

SPECTRUM (DIGITAL)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 1 \text{ kHz}$$

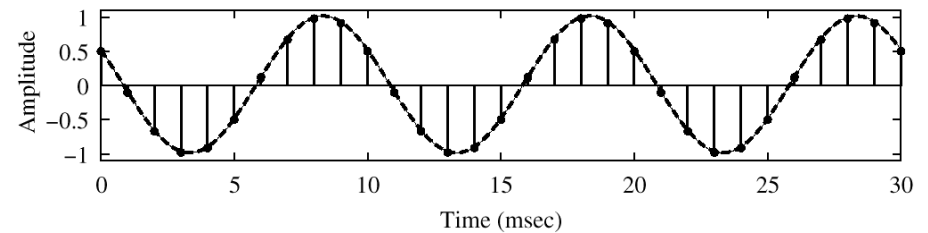
$\frac{1}{2} X^*$
 -0.2π

$\frac{1}{2} X$
 $2\pi(0.1)$

$\hat{\omega}$

$$x[n] = A \cos(2\pi(100)(n/1000) + \phi)$$

100-Hz Cosine Wave: Sampled with $T_s = 1$ msec (1000 Hz)



SPECTRUM (DIGITAL) ???

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 100 \text{ Hz}$$

$\frac{1}{2} X^*$
 -2π

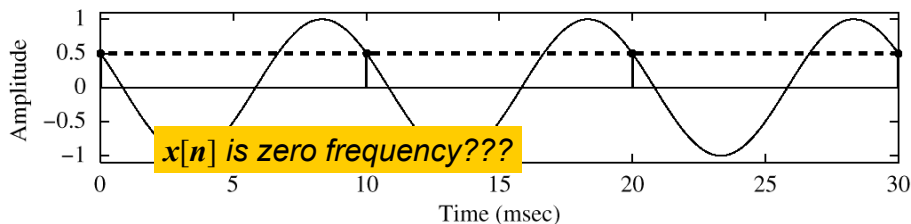
\uparrow ?

$\frac{1}{2} X$
 $2\pi(1)$

$\hat{\omega}$

$$x[n] = A \cos(2\pi(100)(n/100) + \phi)$$

100-Hz Cosine Wave: Sampled with $T_s = 10$ msec (100 Hz)



The REST of the STORY

- Spectrum of $x[n]$ has more than one line for each complex exponential
 - Called ALIASING
 - MANY SPECTRAL LINES
- SPECTRUM is PERIODIC with period = 2π
 - Because

$$A \cos(\hat{\omega}n + \phi) = A \cos((\hat{\omega} + 2\pi)n + \phi)$$

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17

ALIASING DERIVATION

- Other Frequencies give the same $\hat{\omega}$

$$x_1(t) = \cos(400\pi t) \quad \text{sampled at } f_s = 1000 \text{ Hz}$$

$$x_1[n] = \cos(400\pi \frac{n}{1000}) = \cos(0.4\pi n)$$

$$x_2(t) = \cos(2400\pi t) \quad \text{sampled at } f_s = 1000 \text{ Hz}$$

$$x_2[n] = \cos(2400\pi \frac{n}{1000}) = \cos(2.4\pi n)$$

$$x_2[n] = \cos(2.4\pi n) = \cos(0.4\pi n + 2\pi n) = \cos(0.4\pi n)$$

$$\Rightarrow x_2[n] = x_1[n] \quad 2400\pi - 400\pi = 2\pi(1000)$$

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18

ALIASING DERIVATION-2

- Other Frequencies give the same $\hat{\omega}$

$$\text{If } x(t) = A \cos(2\pi(f + \ell f_s)t + \varphi) \quad t \leftarrow \frac{n}{f_s}$$

and we want: $x[n] = A \cos(\hat{\omega}n + \varphi)$

then: $\hat{\omega} = \frac{2\pi(f + \ell f_s)}{f_s} = \frac{2\pi f}{f_s} + \frac{2\pi \ell f_s}{f_s}$

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi \ell$$

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19

ALIASING CONCLUSIONS

- ADDING f_s or $2f_s$ or $-f_s$ to the **FREQ** of $x(t)$ gives exactly the same $x[n]$
 - The samples, $x[n] = x(n/f_s)$ are **EXACTLY THE SAME VALUES**
- GIVEN $x[n]$, WE CAN'T DISTINGUISH f_0 FROM $(f_0 + f_s)$ or $(f_0 + 2f_s)$

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20

NORMALIZED FREQUENCY

- DIGITAL FREQUENCY
Normalized Radian Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi \ell$$

Normalized Cyclic Frequency

$$\hat{f} = \hat{\omega}/(2\pi) = f T_s = f/f_s$$

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21

SPECTRUM for $x[n]$

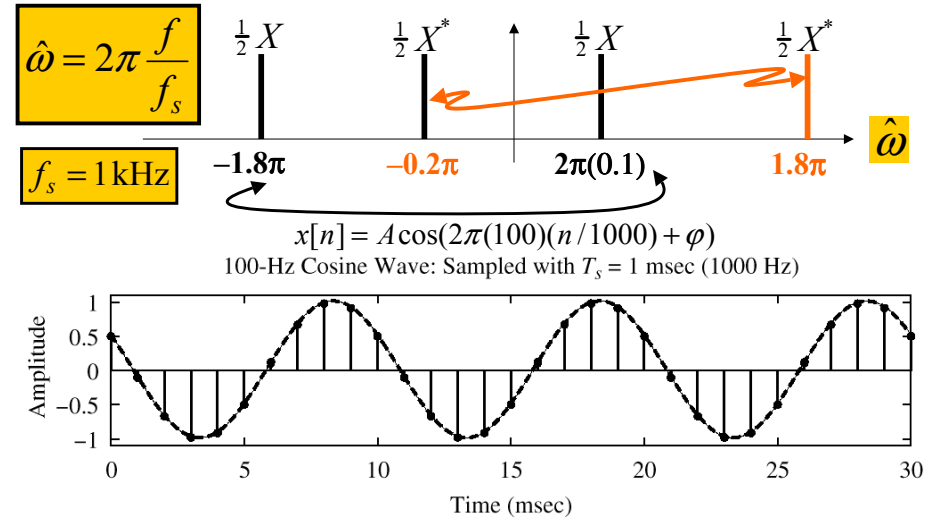
- PLOT versus NORMALIZED FREQUENCY
- INCLUDE **ALL** SPECTRUM LINES
 - ALIASES
 - ADD MULTIPLES of 2π
 - SUBTRACT MULTIPLES of 2π
 - FOLDED ALIASES
 - (to be discussed later)
 - ALIASES of NEGATIVE FREQS

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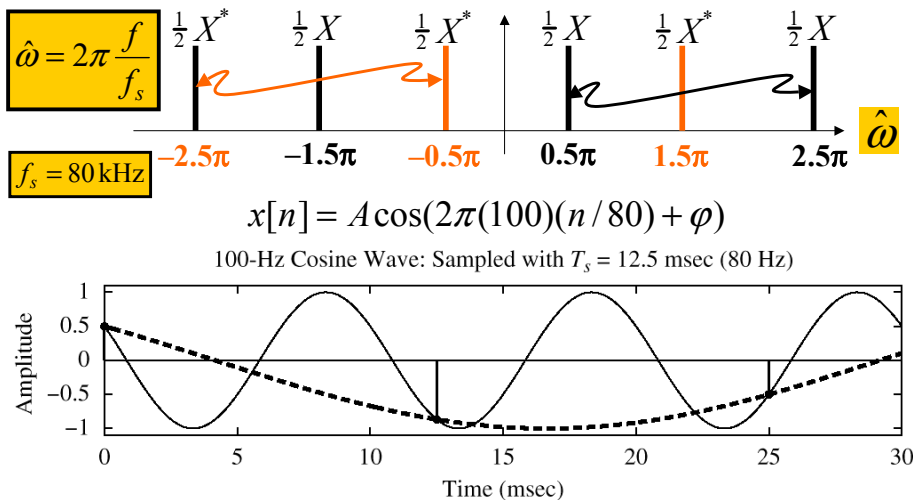
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22

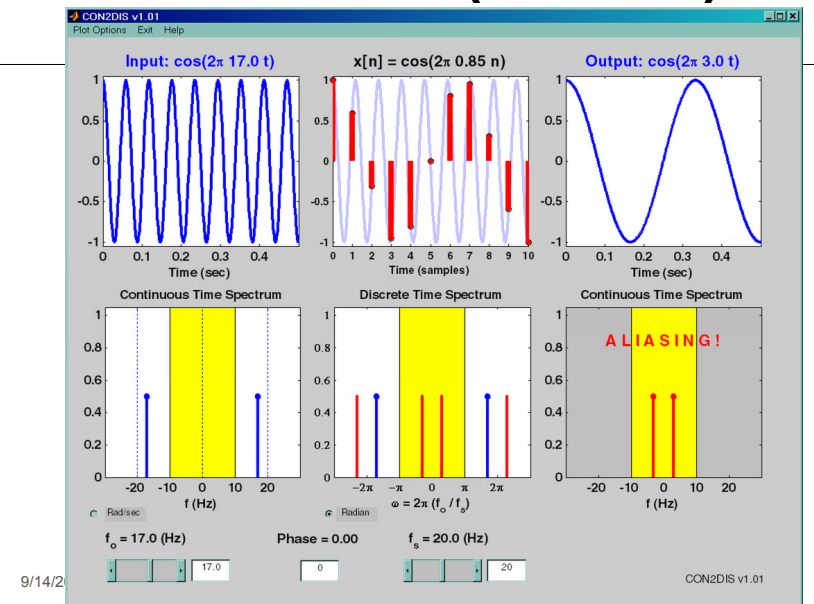
SPECTRUM (MORE LINES)



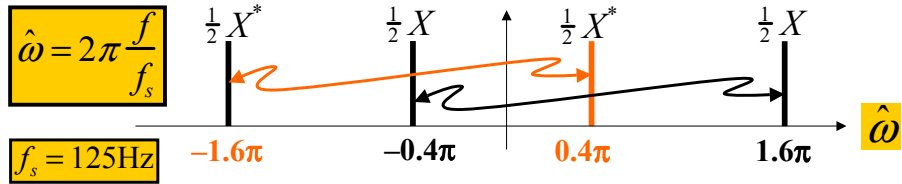
SPECTRUM (ALIASING CASE)



SAMPLING GUI (con2dis)



SPECTRUM (FOLDING CASE)



$$x[n] = A \cos(2\pi(100)(n/125) + \varphi)$$

100-Hz Cosine Wave: Sampled with $T_s = 8$ msec (125 Hz)

