DSP First, 2/e

LECTURE #2
Phase & Time-Shift
Delay & Attenuation

READING ASSIGNMENTS

- This Lecture:
  - Chapter 2, Sects. 2-3 to 2-5
- Appendix A: Complex Numbers
  - Appendix B: MATLAB
  - Next Lecture: Complex Exponentials

LECTURE OBJECTIVES

- Derive Sinusoid **Formula** from a plot
- Relate **TIME-SHIFT** to **PHASE**
- Signal **ENVELOPE** defined
- **ATTENUATION** of Decaying Sinusoid

SINUSOIDAL SIGNAL

$$A\cos(\omega t + \varphi)$$

- **FREQUENCY** ω
  - **Radians/sec**
  - or, Hertz (cycles/sec)

$$\omega = (2\pi) f$$

- **PERIOD** (in sec)

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

- **AMPLITUDE** A
  - Magnitude
- **PHASE** φ

PLOTTING COSINE SIGNAL from the FORMULA

$$5\cos(\underline{0.3\pi} t + 1.2\pi)$$

- Determine **period**:

$$T = 2\pi/\omega = 2\pi/0.3\pi = 20/3$$

- Determine a **peak** location by solving

$$(\omega t + \varphi) = 0$$

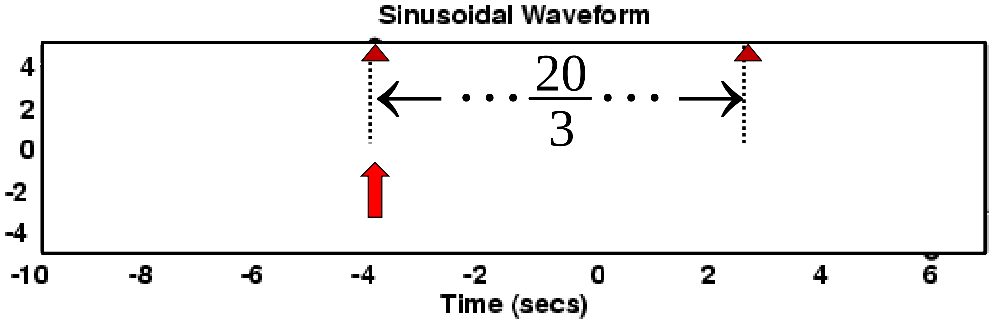
$$0.3\pi t + 1.2\pi = 0$$

- **Peak at t=-4**

ANSWER for the PLOT

$$5\cos(0.3\pi t + 1.2\pi)$$

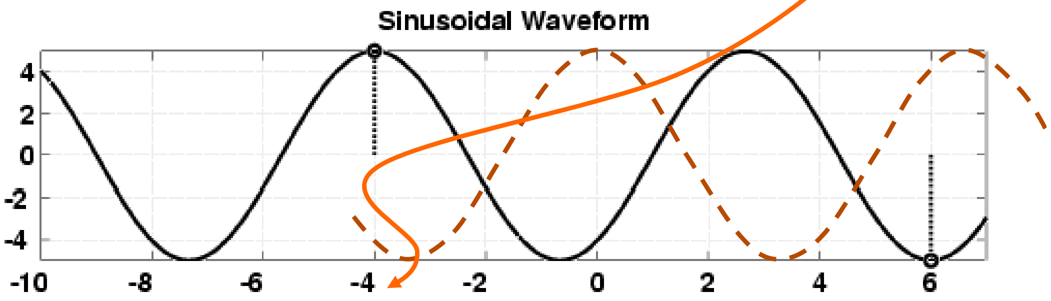
- Use T=20/3 and the peak location at t = -4



TIME-SHIFTED SINUSOID

$$x(t) = 5\cos(0.3\pi t)$$ *One peak at t = 0*

$$x(t+4) = 5\cos(0.3\pi(t+4)) = 5\cos(0.3\pi(t-(-4)))$$



Peak shifts from t=0 to t = -4

TIME-SHIFT

- In a mathematical formula we can replace t with $t-t_m$

$$x(t - t_m) = A\cos(\omega(t - t_m))$$

- Thus the t=0 point moves to $t=t_m$
- Peak value of $\cos(\omega(t-t_m))$ is now at $t=t_m$

PHASE ←→ TIME-SHIFT

- Equate the formulas:

$$A\cos(\omega(t-t_m)) = A\cos(\omega t+\varphi)$$

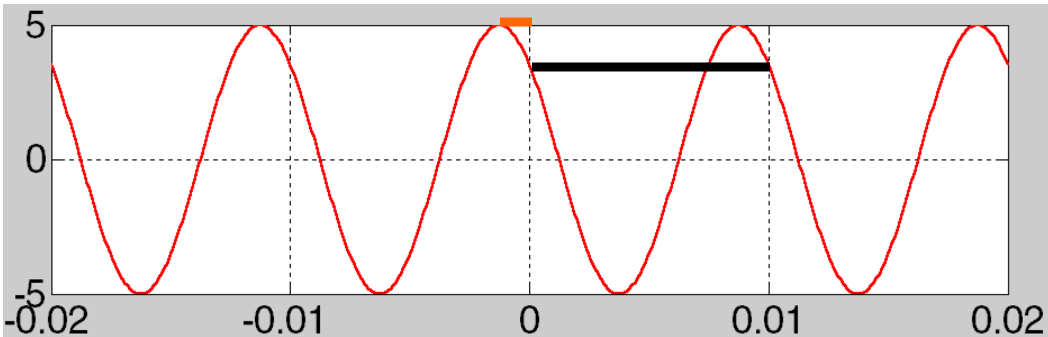
- and we obtain: $-\omega t_m = \varphi$
- or, $t_m = -\dfrac{\varphi}{\omega}$

SINUSOID from a PLOT

3 steps

- Measure the period, T
  - Between peaks or zero crossings
  - Compute frequency: $\omega = 2\pi/T$
- Measure time of a peak: t_m
  - Compute phase: $\phi = -\omega\, t_m$
- Measure height of positive peak: A

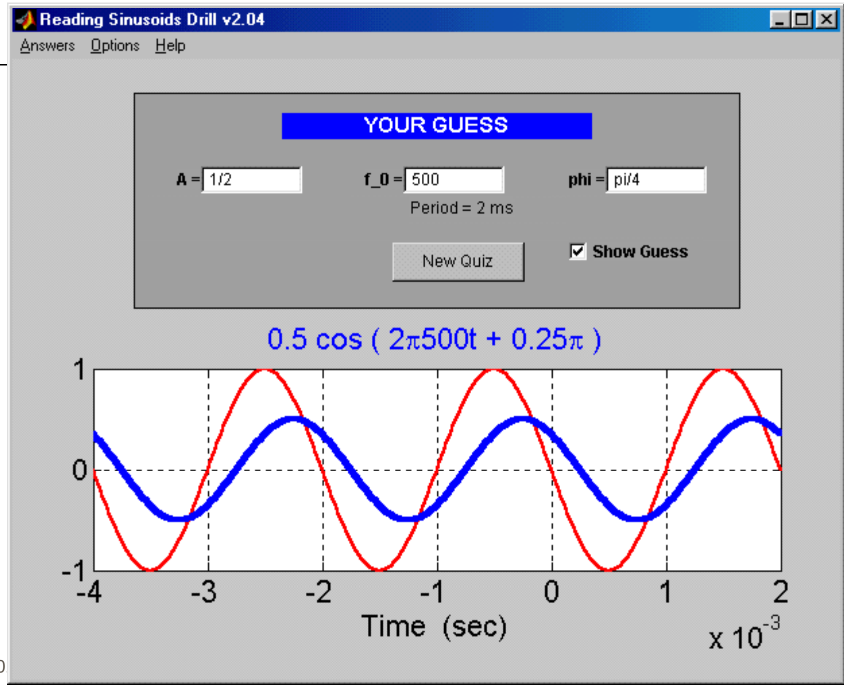
(A, ω, ϕ) from a PLOT



$T = \dfrac{0.01\,\text{sec}}{1\ \text{period}} = \dfrac{1}{100}$ → $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{0.01} = 200\pi$

$t_m = -0.00125\,\text{sec}$ → $\varphi = -\omega t_m = -(200\pi)(t_m) = 0.25\pi$

SINE DRILL (MATLAB GUI)



PHASE is AMBIGUOUS

- The cosine signal is periodic
  - Period is 2π

$$A\cos(\omega t + \varphi + 2\pi) = A\cos(\omega t + \varphi)$$

  - Thus adding any multiple of 2π leaves x(t) unchanged

$$\text{if } \; t_m = \frac{-\varphi}{\omega}, \; \text{then}$$

$$t_{m_2} = \frac{-(\varphi + 2\pi)}{\omega} = \frac{-\varphi}{\omega} - \frac{2\pi}{\omega} = t_m - T$$

PHASE is AMBIGUOUS

- The cosine signal is periodic
  - Period is 2π

$$A\cos(\omega t + \varphi + 2\pi) = A\cos(\omega t + \varphi)$$

  - **Thus adding any multiple of 2π to the phase leaves x(t) unchanged**
  - Equivalent to time-shifting by one period:

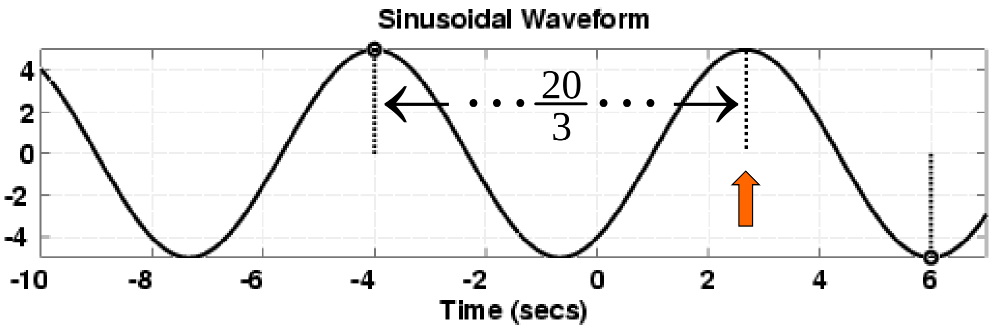
$$A\cos(\omega t + \varphi + 2\pi) =$$

$$A\cos(\omega(t + 2\pi/\omega) + \varphi) = A\cos(\omega(t + T) + \varphi)$$

PLOT the SINUSOID

$$5\cos(0.3\pi t + 1.2\pi) = 5\cos(0.3\pi t - 0.8\pi)$$

- The peak location at t = 8/3 = 2.666



Peak Locations of a Sinusoid Function

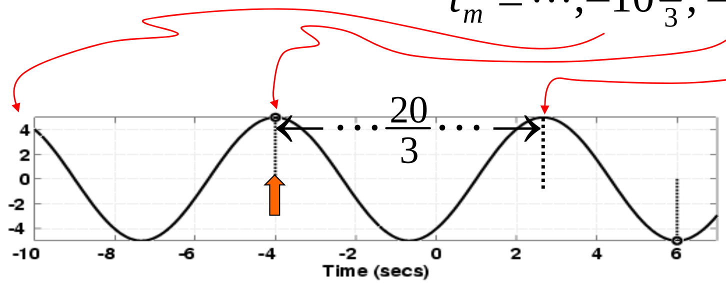
- $\cos(\theta)$ attains max value at

$$\theta = 2n\pi, \; n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$0.3\pi t + 1.2\pi = 2n\pi$$

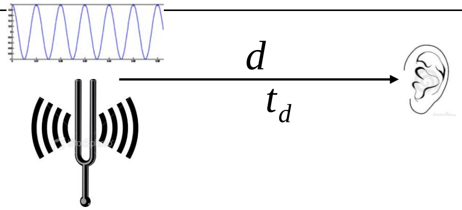
$$5\cos(0.3\pi t + 1.2\pi)$$

$$t_m = \cdots, -10\tfrac{2}{3}, -4, \; 2\tfrac{2}{3}, \cdots$$



Many peaks; One will be closest to the origin

Time Shift and Time Delay



At the origin/source:

$$x_0(t) = A\cos(\omega_0 t + \varphi_0)$$

The distance between origin/source and destination is d; it takes t_d for the wave to travel to the destination:

At the destination: *assuming no attenuation*

$$x_d(t) = A\cos(\omega_0(t - t_d) + \varphi_0) = A\cos(\omega_0 t - \omega_0 t_d + \varphi_0)$$

Phase at the destination: $(-\omega_0 t_d + \varphi_0)_{\mathrm{mod}\,2\pi}$

Example

$$x_o(t) = 5\cos(0.3\pi t + 1.2\pi)$$

Peaks occur at $t_m = \cdots, -10\tfrac{2}{3}, -4, 2\tfrac{2}{3}, \cdots$

Suppose this is a tonal sound that travels at 1000 ft/s.

And suppose the distance is 2000 ft, which means a time delay of 2s. Then,

$$x_d(t) = 5\cos(0.3\pi(t - 2) + 1.2\pi)$$
$$= 5\cos(0.3\pi t - 0.6\pi + 1.2\pi) = 5\cos(0.3\pi t + 0.6\pi)$$

The wave at the destination will have peaks at

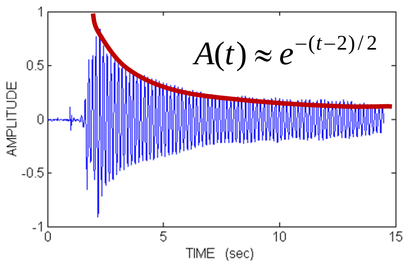
$$t_m = \cdots, -8\tfrac{2}{3}, -2, 4\tfrac{2}{3}, \cdots$$

Attenuation

In real waves, there will always be a certain degree of attenuation, which is the reduction of the signal amplitude over time and/or over distance.

$$x(t) = A\cos(\omega t + \varphi)$$

In a sinusoid, A ***is a constant.***



However, the amplitude can have exponential decay, e.g.,

$$A(t) = Ae^{-t/\alpha}$$

$$x(t) = Ae^{-t/\alpha}\cos(\omega t + \varphi)$$

MATLAB Example (I)

Generating sinusoids in MATLAB is easy:

```
% define how many values in a second
fs = 8000;
% define array tt for time
% time runs from -1s to +3.2s
% sampled at an interval of 1/fs
tt = -1 : 1/fs : 3.2;
xx = 2.1 * cos(2*pi*440*tt + 0.4*pi);
```

The array **xx** *then contains a "sampled" signal of:*

$$x(t) = 2.1\cos(880\pi t + 0.4\pi)$$

MATLAB Example (II)

Introducing attenuation with time

```
% fs defines how many values per second
fs = 8000;
tt = -1 : 1/fs : 3.2;
yy = exp(-abs(tt)*1.2);% exponential decay
yy = xx.*yy;
soundsc(yy,fs)
```

Array **yy** *contains a signal with changing amplitude:*

$$y(t) = 2.1e^{-1.2|t|}\cos(880\pi t + 0.4\pi)$$

Soundsc *lets you hear the signal* **yy**

Plotting the Signal