DSP-First, 2/e

LECTURE #3
Complex Exponentials
& Complex Numbers

READING ASSIGNMENTS

- This Lecture:
  - Chapter 2, Sects. 2-3 to 2-5
- Appendix A: Complex Numbers
  - Appendix B: MATLAB
  - Next Lecture: Complex Exponentials

LECTURE OBJECTIVES

- Introduce more tools for manipulating complex numbers
  - Conjugate
  - Multiplication & Division
  - Powers
  - N-th Roots of unity

$$\text{For } z = e^{j2\pi k/N}, \quad z^N = 1$$

LECTURE OBJECTIVES

- Phasors = Complex Amplitude
  - Complex Numbers **represent** Sinusoids

$$A\cos(\omega t + \varphi) = \Re\{(Ae^{j\varphi})e^{j\omega t}\}$$

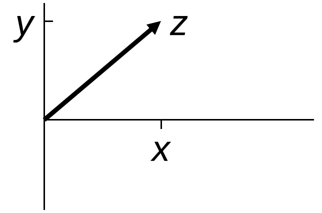
- *Next Lecture: Develop the ABSTRACTION:*
  - *Adding Sinusoids = Complex Addition*
  - ***PHASOR ADDITION THEOREM***

WHY? What do we gain?

- Sinusoids are the basis of DSP,
  - but trig identities are very tedious
- Abstraction of complex numbers
  - Represent cosine functions
  - Can replace most trigonometry with **algebra**
- **Avoid all Trigonometric manipulations**

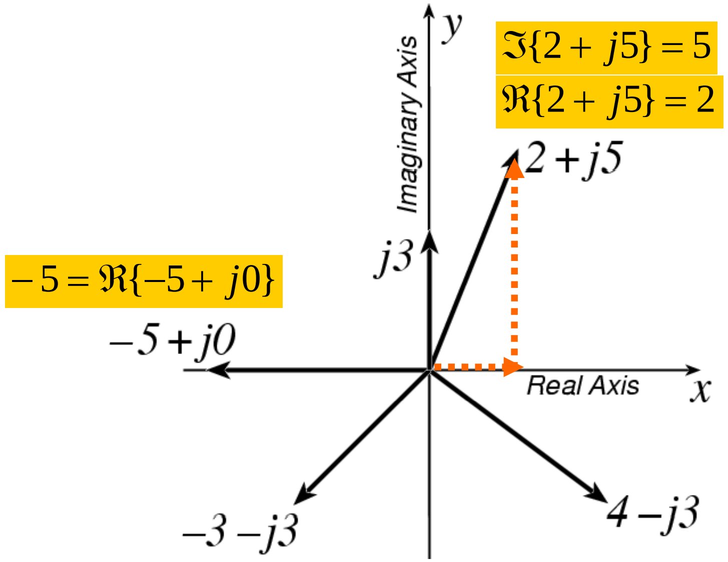
COMPLEX NUMBERS

- To solve: $z^2 = -1$
  - $z = \boldsymbol{j}$
  - Math and Physics use $z = \boldsymbol{i}$
- Complex number: $z = x + \boldsymbol{j}\, y$



Cartesian coordinate system

PLOT COMPLEX NUMBERS



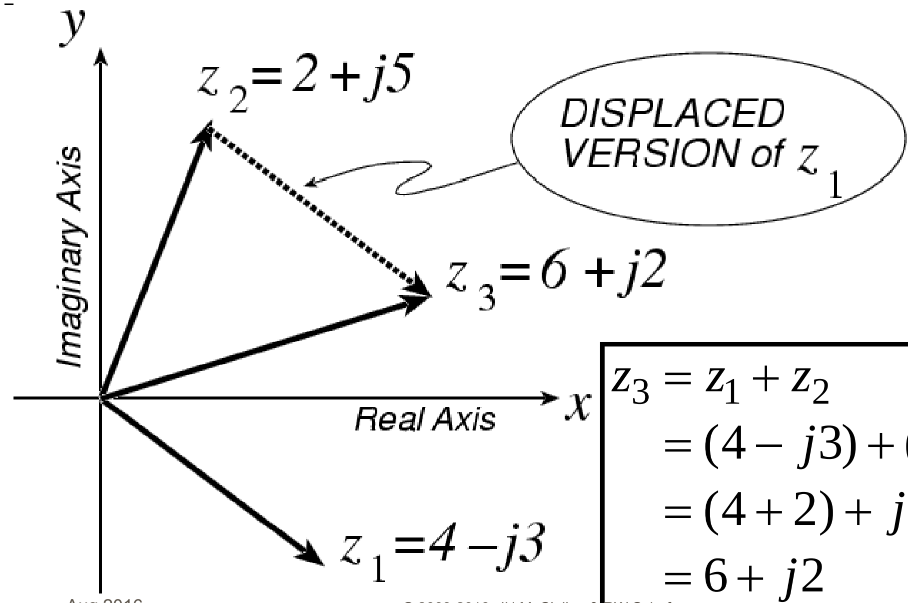
Real part:

$$x = \Re\{z\}$$

Imaginary part:

$$y = \Im\{z\}$$

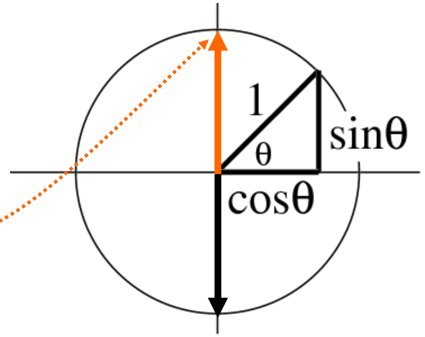
COMPLEX ADDITION = VECTOR Addition



$$\begin{aligned} z_3 &= z_1 + z_2 \\ &= (4 - j3) + (2 + j5) \\ &= (4+2) + j(-3+5) \\ &= 6 + j2 \end{aligned}$$

*** POLAR FORM ***

- **Vector** Form
  - **Length** =1
  - **Angle** = θ
- Common Values
  - **j** has angle of 0.5π
  - –1 has angle of π
  - – **j** has angle of 1.5π
  - also, angle of –**j could** be –0.5π = 1.5π –2π
  - because the PHASE is **AMBIGUOUS**

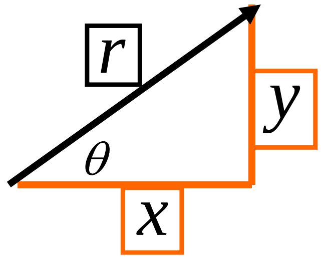


POLAR <--> RECTANGULAR

- Relate (x,y) to (r,θ)

$$r^2 = x^2 + y^2$$

$$\theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right)$$



Most calculators do Polar-Rectangular

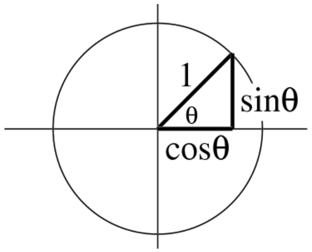
$$x = r\cos\theta$$

$$y = r\sin\theta$$

Need a notation for POLAR FORM

Euler's FORMULA

- **Complex Exponential**
  - **Real part is cosine**
  - Imaginary part is sine
  - Magnitude is one

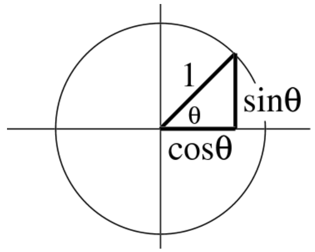


$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

Cosine = Real Part

- Complex Exponential
  - **Real part is cosine**
  - Imaginary part is sine



$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

$$\Re\{re^{j\theta}\} = r\cos(\theta)$$

Common Values of exp(jθ)

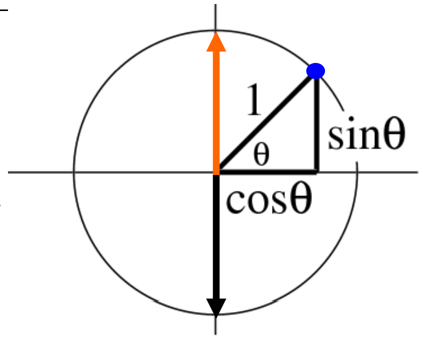
- Changing the angle

$$\theta = 0 \rightarrow 1 = 1 + j0 = e^{j0} = e^{j2n\pi}$$

$$\theta = \pi \rightarrow -1 = -1 + j0 = e^{j\pi} = e^{j(2n+1)\pi}$$

$$\theta = \pi/2 \rightarrow j = e^{j\pi/2} = e^{j(2n+1/2)\pi}$$

$$\theta = 3\pi/2 \rightarrow -j = e^{j3\pi/2} = e^{-j\pi/2} = e^{j(2n-1/2)\pi}$$

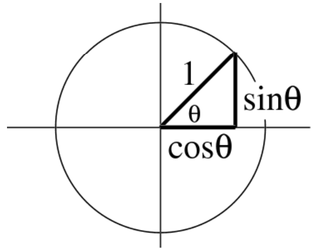


$$1 \pm j = \sqrt{2}e^{\pm j\pi/4} \qquad \pm 1 + j = ?$$

COMPLEX EXPONENTIAL

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- Interpret this as a **Rotating Vector**
  - $\theta = \omega t$
  - Angle changes vs. time
  - ex: $\omega = 20\pi$ rad/s
  - Rotates 0.2π in 0.01 secs



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Cos = REAL PART

Real Part of Euler's

$$\cos(\omega t) = \Re\{e^{j\omega t}\}$$

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi)$$

So,

$$A\cos(\omega t + \varphi) = \Re\{Ae^{j(\omega t + \varphi)}\}$$

$$= \Re\{Ae^{j\varphi}e^{j\omega t}\}$$

COMPLEX AMPLITUDE

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi) = \Re\{Ae^{j\varphi}e^{j\omega t}\}$$

Sinusoid = REAL PART of complex exp: z(t)=($Ae^{j\phi}$)$e^{j\omega t}$

$$x(t) = \Re\{Xe^{j\omega t}\} = \Re\{z(t)\}$$

Complex AMPLITUDE = X, which is a constant

$$X = Ae^{j\varphi} \qquad \text{when } z(t) = Xe^{j\omega t}$$

POP QUIZ: Complex Amp

- Find the COMPLEX AMPLITUDE for:

$$x(t) = \sqrt{3}\cos(77\pi t + 0.5\pi)$$

- Use EULER's FORMULA:

$$x(t) = \Re\{\sqrt{3}e^{j(77\pi t + 0.5\pi)}\}$$
$$= \Re\{\sqrt{3}e^{j0.5\pi}e^{j77\pi t}\}$$

$$X = \sqrt{3}e^{j0.5\pi}$$

POP QUIZ-2: Complex Amp

- Determine the 60-Hz sinusoid whose COMPLEX AMPLITUDE is: $X = \sqrt{3} + j3$
- Convert ***X*** to **POLAR**:

$$x(t) = \Re\{(\sqrt{3} + j3)e^{j(120\pi t)}\}$$
$$= \Re\{\sqrt{12}e^{j\pi/3}e^{j120\pi t}\}$$

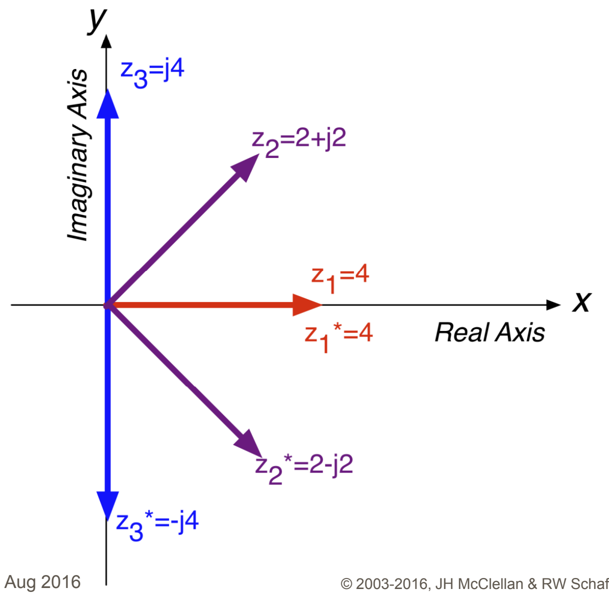
$$\Rightarrow x(t) = \sqrt{12}\cos(120\pi t + \pi/3)$$

COMPLEX CONJUGATE (z*)

- Useful concept: change the sign of **all j's**
- **RECTANGULAR**: If z = x + ***j*** y, then the complex conjugate is z* = x – ***j*** y
- **POLAR**: Magnitude is the same but angle has sign change

$$z = re^{j\theta} \Rightarrow z^* = re^{-j\theta}$$

COMPLEX CONJUGATION



- *Flips vector about the real axis!*

USES OF CONJUGATION

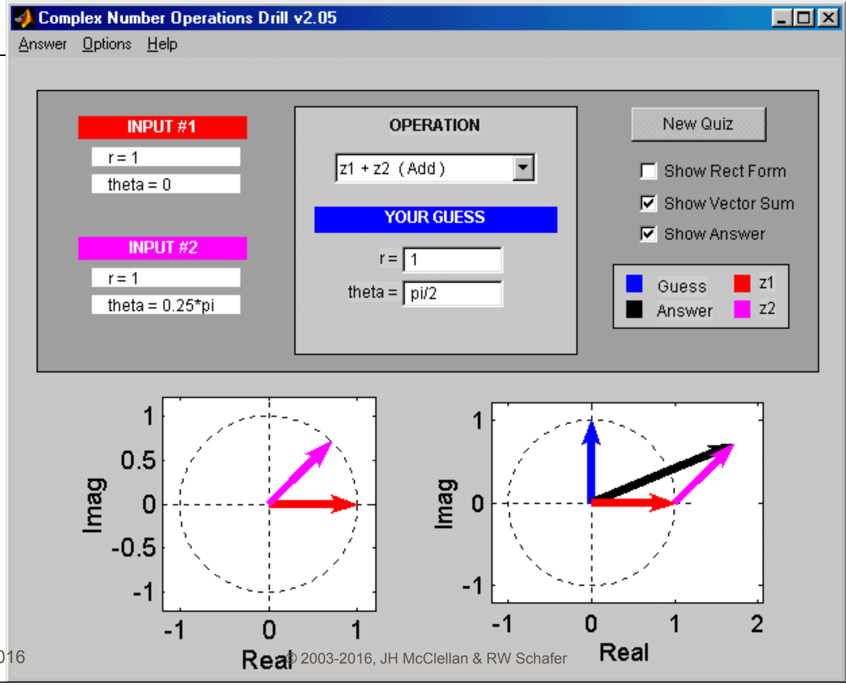
- Conjugates useful for many calculations
- Real part:

$$\frac{z + z^*}{2} = \frac{(x + jy) + (x - jy)}{2} = x = \Re\{z\}$$

- Imaginary part:

$$\frac{z - z^*}{2j} = \frac{j2y}{2j} = y = \Im\{z\}$$

Z DRILL (Complex Arith)



Inverse Euler Relations

- Cosine is real part of exp, sine is imaginary part
- Real part: $\frac{z + z^*}{2} = \Re\{z\}$

$$z = e^{j\theta}, \Rightarrow \Re\{e^{j\theta}\} = \frac{e^{j\theta} + e^{-j\theta}}{2} = \cos(\theta)$$

- Imaginary part:

$$\frac{z - z^*}{2j} = y = \Im\{z\}$$

$$z = e^{j\theta}, \Rightarrow \Im\{e^{j\theta}\} = \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin(\theta)$$

Mag & Magnitude Squared

- Magnitude Squared (polar form):

$$z z^* = (re^{j\theta})(re^{-j\theta}) = r^2 = |z|^2$$

- Magnitude Squared (Cartesian form):

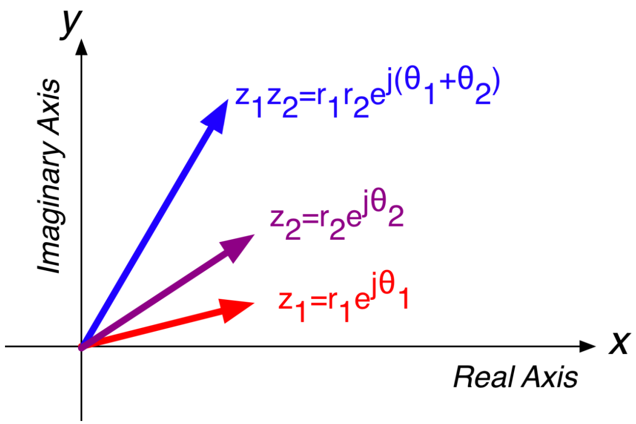
$$z z^* = (x + jy) \times (x - jy) = x^2 - j^2 y^2 = x^2 + y^2$$

- Magnitude of complex exponential is one:

$$|e^{j\theta}|^2 = \cos^2(\theta) + \sin^2(\theta) = 1$$

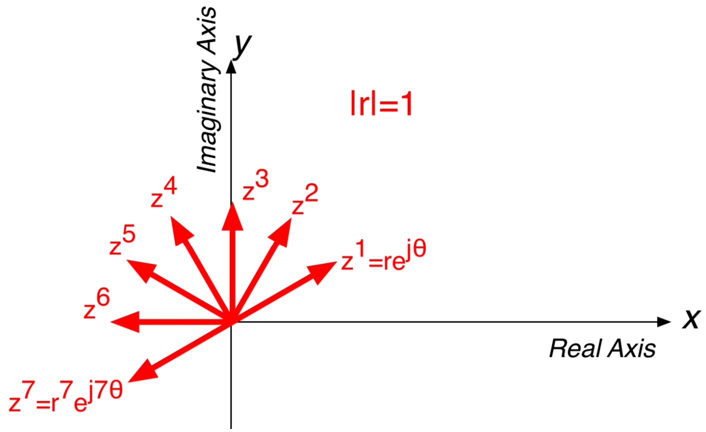
COMPLEX MULTIPLY = VECTOR ROTATION

- Multiplication/division scales and rotates vectors

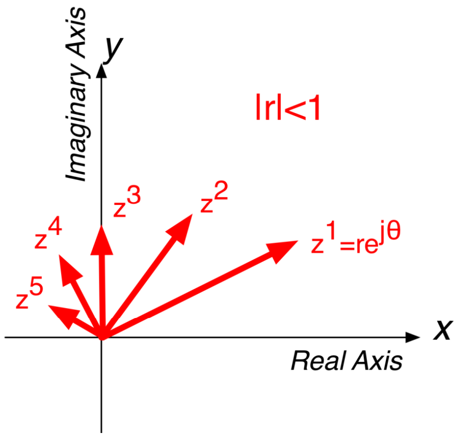


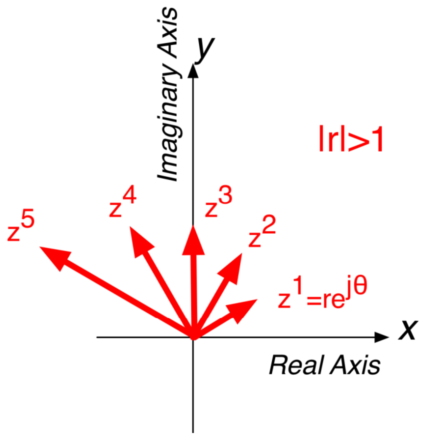
POWERS

- Raising to a power N rotates vector by Nθ and scales vector length by r^N



MORE POWERS





ROOTS OF UNITY

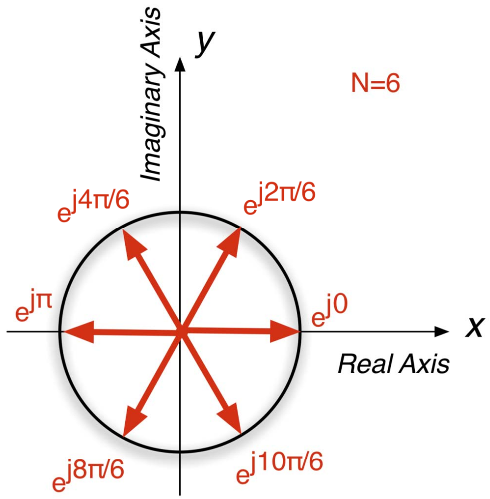
- We often have to solve $z^N = 1$
- How many solutions?

$$z^N = r^N e^{jN\theta} = 1 = e^{j2\pi k}$$

$$\Rightarrow r = 1, \quad N\theta = 2\pi k \Rightarrow \theta = \frac{2\pi k}{N}$$

$$z = e^{j2\pi k/N}, \quad k = 0, 1, 2, \ldots N-1$$

ROOTS OF UNITY for N=6



- Solutions to $z^N=1$ are N equally spaced vectors on the unit circle!
- What happens if we take the sum of all of them?

Sum the Roots of Unity

- Looks like the answer is zero (for N=6)

$$\sum_{k=0}^{N-1} e^{j2\pi k/N} = 0?$$

- Write as geometric sum

$$\sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r} \quad \text{then let } r = e^{j2\pi/N}$$

$$\text{Numerator} \quad 1-r^N = 1-(e^{j2\pi/N})^N = 1-e^{j2\pi} = 0$$

Integrate Complex Exp

- Needed later to describe periodic signals in terms of sinusoids (Fourier Series)
  - Especially over one period

$$\int_a^b e^{j\theta} d\theta = \left.\frac{e^{j\theta}}{j}\right|_a^b = \frac{e^{jb}-e^{ja}}{j}$$

$$\int_0^T e^{j2\pi t/T} dt = \frac{e^{j2\pi T/T} - e^{j0}}{j} = \frac{1-1}{j} = 0$$

BOTTOM LINE

- **CARTESIAN**: Addition/subtraction is most efficient in Cartesian form
- **POLAR**: good for multiplication/division
- STEPS:
  - Identify arithmetic operation
  - Convert to easy form
  - Calculate
  - Convert back to original form