DSP-First, 2/e

LECTURE #4

Phasor Addition Theorem

READING ASSIGNMENTS

- This Lecture:
  - Chapter 2, Section 2-6
- Other Reading:
  - Appendix A: Complex Numbers
    - Appendix B: MATLAB
    - Next Lecture: start Chapter 3

LECTURE OBJECTIVES

- Phasors = Complex Amplitude
  - Complex Numbers **represent** Sinusoids

$$A\cos(\omega t+\varphi)=\Re\{(Ae^{j\varphi})e^{j\omega t}\}$$

- *Develop the ABSTRACTION:*
  - *Adding Sinusoids = Complex Addition*
  - ***PHASOR ADDITION THEOREM***

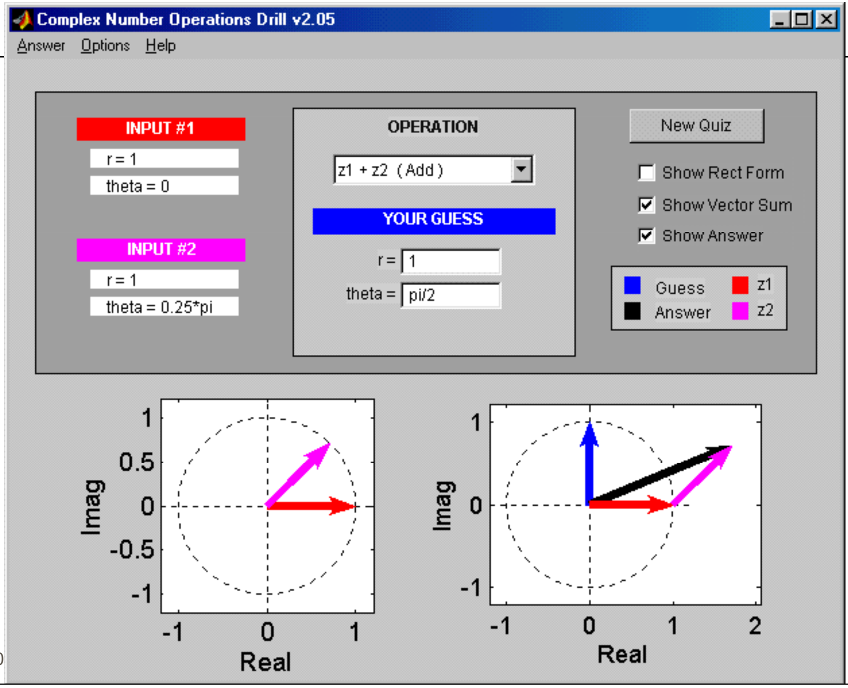
Adding Complex Numbers

- Polar Form
  - Could convert to Cartesian and back out
  - **Use Calculator that does complex ops !**
  - Use MATLAB
  - Visualize the vectors

$$1.7e^{j70\pi/180}+1.9e^{j200\pi/180}=Ae^{j\varphi}\ \ ?$$

$$1.532e^{j141.79\pi/180}$$

Z DRILL (Complex Arith)



Cos = REAL PART

What about sinusoidal signals over time?
Real part of Euler's

$$\cos(\omega t) = \Re\{e^{j\omega t}\}$$

General Sinusoid

$$A\cos(\omega t + \varphi) = \Re\{Ae^{j(\omega t+\varphi)}\}$$
$$= \Re\{Ae^{j\varphi}e^{j\omega t}\}$$

***Complex Amplitude**: Constant* *Varies with time*

POP QUIZ: Complex Amp

- Find the COMPLEX AMPLITUDE for:

$$x(t) = \sqrt{3}\cos(77\pi t + 0.5\pi)$$

- Use EULER's FORMULA:

$$x(t) = \Re\{\sqrt{3}e^{j(77\pi t + 0.5\pi)}\}$$
$$= \Re\{\sqrt{3}e^{j0.5\pi}e^{j77\pi t}\}$$

$$X = \sqrt{3}e^{j0.5\pi}$$

POP QUIZ-2: Complex Amp

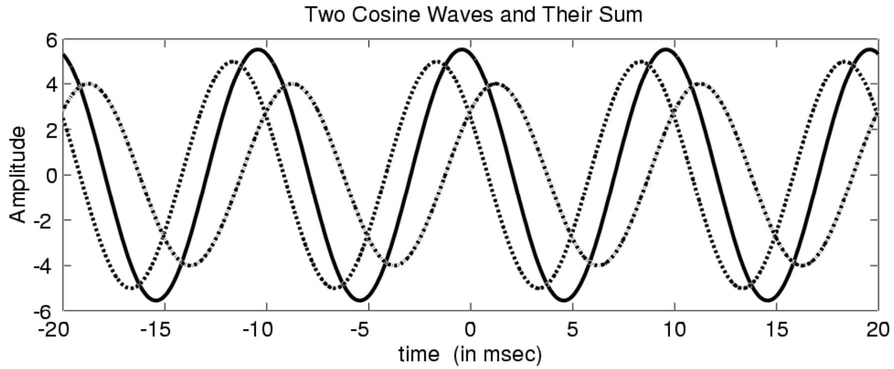
- Determine the 60-Hz sinusoid whose COMPLEX AMPLITUDE is: $X = \sqrt{3} + j3$
- Convert ***X*** to **POLAR**:

$$x(t) = \Re\{(\sqrt{3} + j3)e^{j(120\pi t)}\}$$
$$= \Re\{\sqrt{12}e^{j\pi/3}e^{j120\pi t}\}$$

$$\Rightarrow x(t) = \sqrt{12}\cos(120\pi t + \pi/3)$$

WANT to ADD SINUSOIDS

- **Main point to remember**: Adding sinusoids of common frequency results in sinusoid with **SAME** frequency



PHASOR ADDITION RULE

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \varphi_k)$$

$$= A\cos(\omega_0 t + \varphi)$$

Get the new complex amplitude by complex addition

$$\sum_{k=1}^{N} A_k e^{j\varphi_k} = Ae^{j\varphi}$$

Phasor Addition Proof

$$\sum_{k=1}^{N} A_k \cos(\omega_0 t + \varphi_k) = \sum_{k=1}^{N} \Re\{A_k e^{j(\omega_0 t + \varphi_k)}\}$$

$$= \Re\left\{\sum_{k=1}^{N} A_k e^{j\varphi_k} e^{j\omega_0 t}\right\}$$

$$= \Re\left\{\left(\sum_{k=1}^{N} A_k e^{j\varphi_k}\right) e^{j\omega_0 t}\right\}$$

$$= \Re\left\{\left(Ae^{j\varphi}\right) e^{j\omega_0 t}\right\} = A\cos(\omega_0 t + \varphi)$$

POP QUIZ: Add Sinusoids

- ADD THESE 2 SINUSOIDS:

$$x_1(t) = \cos(77\pi t - \pi)$$

$$x_2(t) = \sqrt{3}\cos(77\pi t + 0.5\pi)$$

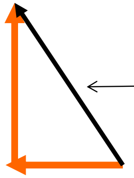
- COMPLEX (PHASOR) ADDITION:

$$1e^{-j\pi} + \sqrt{3}e^{j0.5\pi}$$

POP QUIZ (answer)

- COMPLEX ADDITION: $1e^{-j\pi} + \sqrt{3}e^{j0.5\pi}$

$\sqrt{3}e^{j\pi/2} = j\sqrt{3}$



$-1 + j\sqrt{3} = 2e^{j2\pi/3}$

$e^{-j\pi} = -1$

- CONVERT back to cosine form:

$$x_3(t) = 2\cos(77\pi t + \tfrac{2\pi}{3})$$

ADD SINUSOIDS EXAMPLE

- ALL SINUSOIDS have SAME FREQUENCY
- HOW to GET {Amp,Phase} of RESULT ?

$$x_1(t) = 1.7\cos(2\pi(10)t + 70\pi/180)$$
$$x_2(t) = 1.9\cos(2\pi(10)t + 200\pi/180)$$

$$x_3(t) = x_1(t) + x_2(t) = A\cos(\omega t + \varphi)$$
$$= \Re\{Ae^{j\varphi}e^{j20\pi t}\}$$

Convert Sinusoids to Phasors

- Each sinusoid → Complex Amp

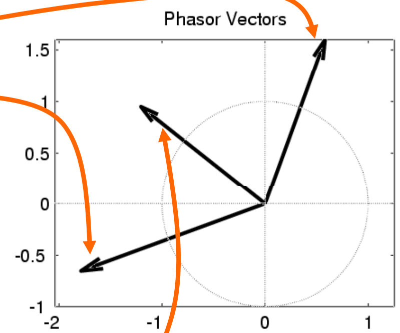
$$1.7\cos(20\pi t + 70\pi/180) \rightarrow 1.7e^{j70\pi/180}$$
$$1.9\cos(20\pi t + 200\pi/180) \rightarrow 1.9e^{j200\pi/180}$$

$$1.7e^{j70\pi/180} + 1.9e^{j200\pi/180} = ?$$
$$1.532e^{j141.79\pi/180}$$

$$\rightarrow 1.532\cos(20\pi t + 141.79\pi/180)$$

Phasor Add: Numerical

- Convert Polar to Cartesian
  - X_1 = 0.5814 + j1.597
  - X_2 = -1.785 - j0.6498
  - sum =
  - X_3 = -1.204 + j0.9476

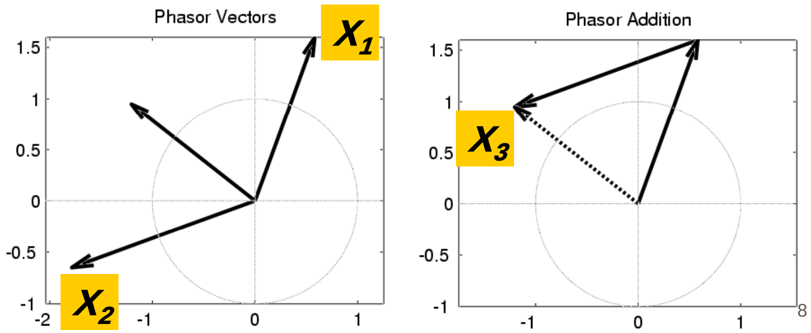


- Convert back to Polar
  - X_3 = 1.532 at angle 141.79π/180
  - This is the sum

ADDING SINUSOIDS IS COMPLEX ADDITION

$$x_1(t) = 1.7\cos(2\pi(10)t + 70\pi/180)$$
$$x_2(t) = 1.9\cos(2\pi(10)t + 200\pi/180)$$
$$x_3(t) = x_1(t) + x_2(t)$$
$$= 1.532\cos(2\pi(10)t + 141.79\pi/180)$$

VECTOR (PHASOR) ADD



Add 20 Sinusoids (MATLAB)

$$s(t) = \sum_{k=1}^{20} \sqrt{k}\cos(120\pi(t - 0.002k))$$

- Each sinusoid → Complex Amp

$$S = Ae^{j\varphi} = \sum_{k=1}^{20} \sqrt{k}\, e^{j120\pi(-0.002k)}$$

- *MATLAB*

```
kk=1:20;
SS = sum( sqrt(kk) .* exp(120i*pi*(-0.002)*kk) );
zprint( SS )
```

$$A = 6.949, \quad \varphi = -1.545$$

Simultaneous Equations-1

- Sum of 3 sinusoids is zero
- Difference of first two is a cosine
- Sum of first and third is a sine
- All three have the same frequency

$$A_1\cos(\omega t + \varphi_1) + A_2\cos(\omega t + \varphi_2) + A_3\cos(\omega t + \varphi_3) = 0$$
$$A_1\cos(\omega t + \varphi_1) - A_2\cos(\omega t + \varphi_2) = \cos(\omega t)$$
$$A_1\cos(\omega t + \varphi_1) + A_3\cos(\omega t + \varphi_3) = \sin(\omega t) = \cos(\omega t - \pi/2)$$

Simultaneous Equations-2

- Each sinusoid → Complex Amp

$A_1\cos(\omega t + \varphi_1) \rightarrow A_1 e^{j\varphi_1}$, call this z_1, etc.

$$\begin{aligned} A_1 e^{j\varphi_1} + A_2 e^{j\varphi_1} + A_3 e^{j\varphi_3} &= 0 \\ A_1 e^{j\varphi_1} - A_2 e^{j\varphi_2} + 0 &= e^{j0} \\ A_1 e^{j\varphi_1} + 0 + A_3 e^{j\varphi_3} &= e^{-j\pi/2} \end{aligned}$$

$$\begin{aligned} z_1 + z_2 + z_3 &= 0 \\ z_1 - z_2 + 0 &= 1 \\ z_1 + 0 + z_3 &= -j \end{aligned}$$

Solve 3 equations in 3 unknowns →

$$\begin{aligned} z_2 &= j = e^{j\pi/2} \\ z_1 &= 1 + j = \sqrt{2}\, e^{j\pi/4} \\ z_3 &= -j - 1 - j = \sqrt{5}\, e^{-j2.034} \end{aligned}$$

Simultaneous Complex Equations

- Write as a matrix:

$$\begin{aligned} z_1 + z_2 + z_3 &= 0 \\ z_1 - z_2 + 0 &= 1 \\ z_1 + 0 + z_3 &= -j \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -j \end{bmatrix}$$

- *MATLAB with backslash operator*

```
Zans = [1,1,1;1,-1,0;1,0,1] \ [0;1;-j]
```