# DSP First, 2/e

Lecture 5

Spectrum Representation

# READING ASSIGNMENTS

- This Lecture:
  - Chapter 3, Section 3-1
- Other Reading:
  - Appendix A: Complex Numbers



# LECTURE OBJECTIVES

- Sinusoids with DIFFERENT Frequencies
  - SYNTHESIZE by Adding Sinusoids

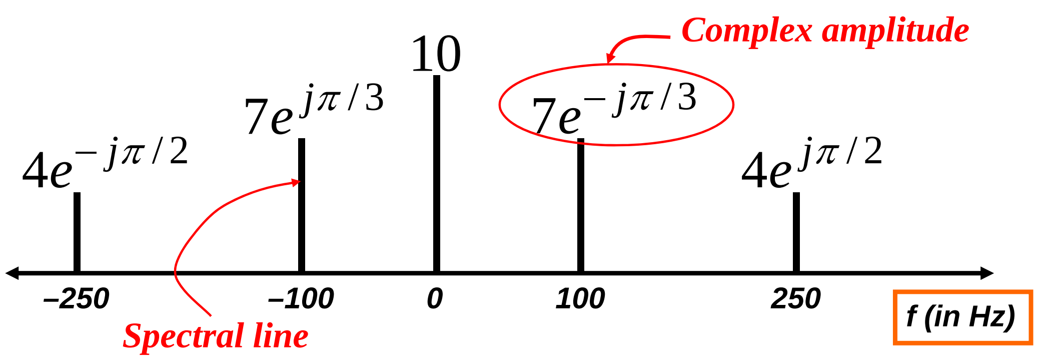
$$x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

- SPECTRUM Representation
  - Graphical Form shows **DIFFERENT** Freqs



# FREQUENCY DIAGRAM

- Want to visualize relationship between frequencies, amplitudes and phases
- Plot Complex Amplitude vs. Frequency



# Another FREQ. Diagram

*Frequency is the vertical axis*


**Figure 3.18** Sheet-music notation is a time–frequency diagram.

***Time is the horizontal axis***

*A musical scale consists of a discrete set of frequencies.*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# MOTIVATION

- Synthesize Complicated Signals
  - Musical Notes
    - Piano uses 3 strings for many notes
    - Chords: play several notes simultaneously
  - Human Speech
    - Vowels have dominant frequencies
    - Application: computer generated speech
  - Can all signals be generated this way?
    - Sum of sinusoids?


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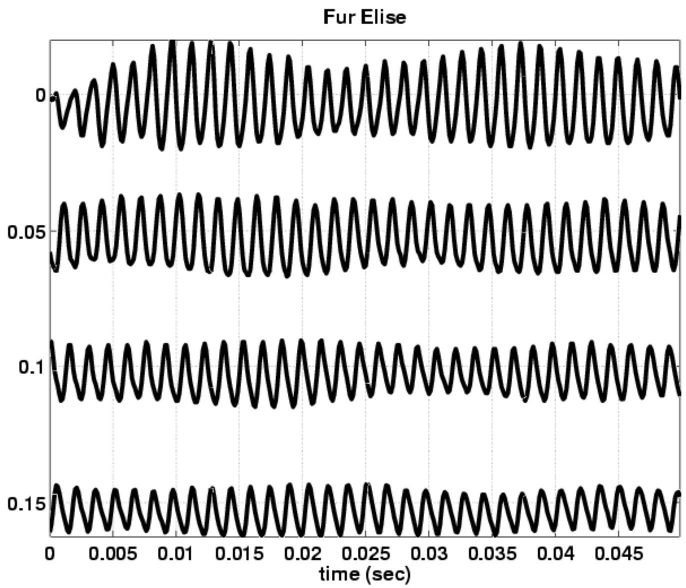
# Fur Elise WAVEFORM

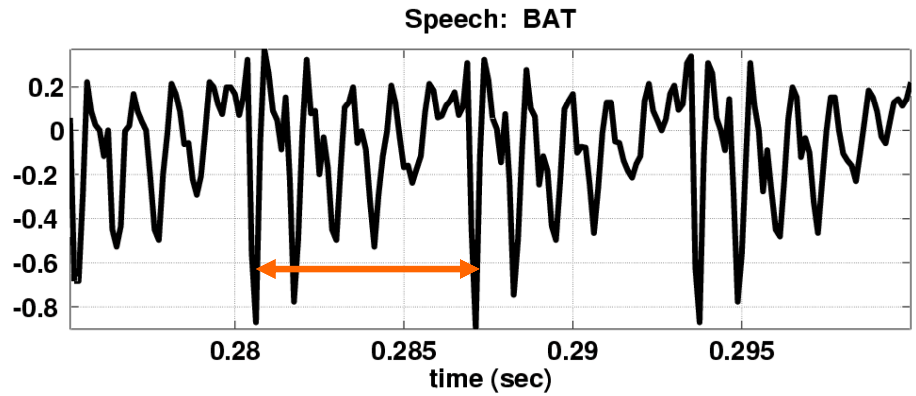



***Beat Notes***




# Speech Signal: BAT

- Nearly **Periodic** in Vowel Region
  - Period is (Approximately) T = 0.0065 sec





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## Euler's Formula Reversed

- Solve for cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

$$\boxed{\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})}$$


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## INVERSE Euler's Formula

- What is the "spectrum" representation for a single sinusoid?
- Solve Euler's formula for cosine (or sine)

$$\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$


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## SPECTRUM Interpretation

- Cosine = sum of 2 complex exponentials:

$$A\cos(7t) = \frac{A}{2}e^{j7t} + \frac{A}{2}e^{-j7t}$$

  - One has a positive frequency
  - The other has negative freq.
  - Amplitude of each is half as big


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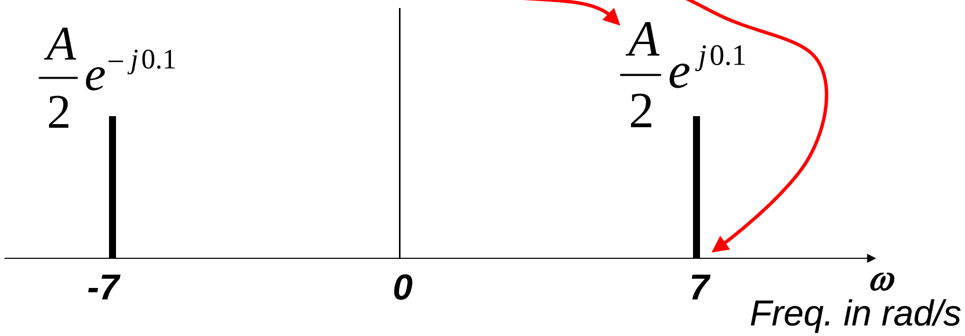
## GRAPHICAL SPECTRUM

$$A\cos(7t + 0.1) = \tfrac{A}{2}e^{j0.1}e^{j7t} + \tfrac{A}{2}e^{-j0.1}e^{-j7t}$$




AMPLITUDE, PHASE & FREQUENCY are labels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# NEGATIVE FREQUENCY

- Is negative frequency real?
- Doppler Radar provides intuition
  - Police radar measures speed by using the Doppler shift principle
  - Let's assume 400Hz ←→60 mph
  - +400Hz means towards the radar
  - -400Hz means away (opposite direction)
  - Think of a train whistle


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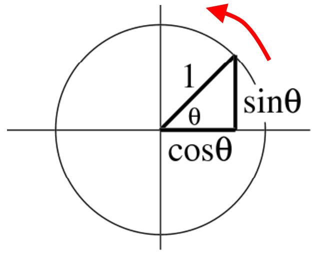
# Negative Frequency is still a rotating phasor

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- View as vector rotating counterclockwise
  - $\theta = \omega t$
  - Angle changes vs. time




$$e^{-j\omega t} = e^{j(-\omega)t}$$

*Negative frequency ➔ clockwise rotation*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# General form of sinusoid spectrum

- General form:

$$A\cos(\omega t + \varphi)$$

$$= \tfrac{A}{2} e^{j\varphi} e^{j\omega t} + \tfrac{A}{2} e^{-j\varphi} e^{-j\omega t}$$

  - Amplitudes are multiplied by ½
  - Complex amplitudes are complex conjugates
    - Called ***conjugate symmetry***


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# SPECTRUM Interpretation

- Cosine = sum of 2 complex exponentials:

$$A\cos(7t + 0.1) = \Re\left\{A e^{j0.1} e^{j7t}\right\}$$

$$= \tfrac{A}{2} e^{j0.1} e^{j7t} + \tfrac{A}{2} e^{-j0.1} e^{-j7t}$$

  - One has a positive frequency
  - The other has negative freq.
  - Amplitude of each is half as big


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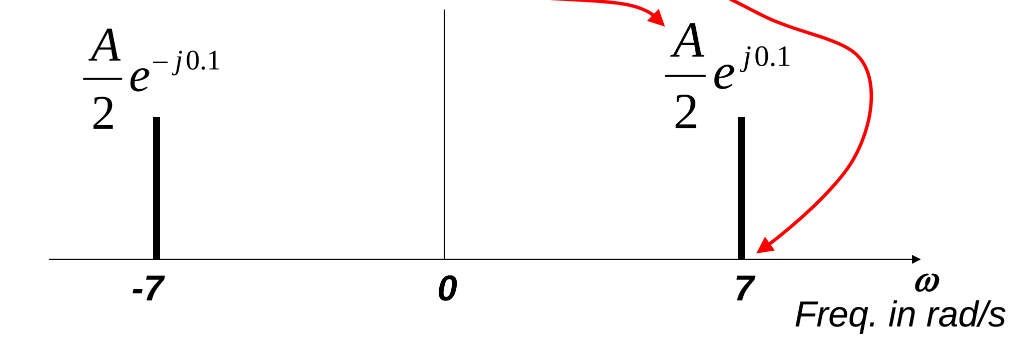
## Recall SPECTRUM of cosine

$$A\cos(7t+0.1) = \frac{A}{2}e^{j0.1}e^{j7t} + \frac{A}{2}e^{-j0.1}e^{-j7t}$$

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## REPRESENTATION of SINE

- Sine = sum of 2 complex exponentials:

$$A\sin(7t) = \frac{A}{2j}e^{j7t} - \frac{A}{2j}e^{-j7t}$$

$$= \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

$$\frac{-1}{j} = j = e^{j0.5\pi}$$

- Positive freq. has phase = -0.5π
- Negative freq. has phase = +0.5π


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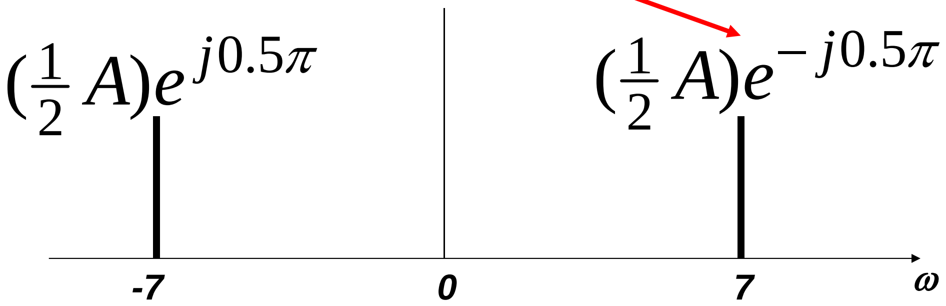
## GRAPHICAL Spectrum of sine

*EXAMPLE of SINE **(has Phase of –π/2)***

$$A\sin(7t) = \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

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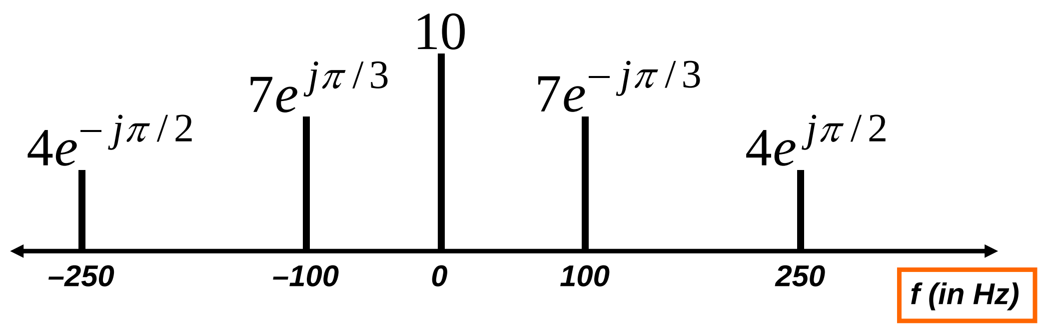
## SPECTRUM ---> SINUSOID

- Add the spectrum components:




***What is the formula for the signal x(t)?***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# Gather (A,ω,φ) information

- Frequencies:
  - -250 Hz
  - -100 Hz
  - **0** Hz
  - 100 Hz
  - 250 Hz
- Amplitude & Phase
  - 4 -π/2
  - 7 +π/3
  - 10 **0**
  - 7 -π/3
  - 4 +π/2

*Note the **conjugate phase***

***DC** is another name for zero-freq component*

***DC** component always has $\phi = 0$ or $\pi$ (for real **x(t)** )*



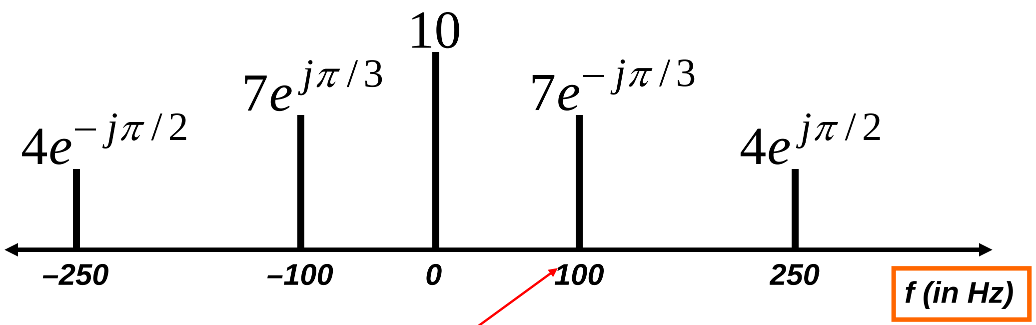
# Add Spectrum Components-1

- **Frequencies:**
  - **-250 Hz**
  - **-100 Hz**
  - **0 Hz**
  - **100 Hz**
  - **250 Hz**
- **Amplitude & Phase**
  - **4 -π/2**
  - **7 +π/3**
  - **10 0**
  - **7 -π/3**
  - **4 +π/2**

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$



# Add Spectrum Components-2




$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$



# Simplify Components

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

***Use Euler's Formula to get REAL sinusoids:***

$$A\cos(\omega t + \varphi) = \tfrac{1}{2}Ae^{j\varphi}e^{j\omega t} + \tfrac{1}{2}Ae^{-j\varphi}e^{-j\omega t}$$

# FINAL ANSWER

$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

***So, we get the general form:***

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$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# Example: Synthetic Vowel

- Sum of 5 Frequency Components

| $f_k$ (Hz) | $X_k$ | Mag | Phase (rad) |
|---|---|---|---|
| 200 | $(771 + j12202)$ | 12,226 | 1.508 |
| 400 | $(-8865 + j28048)$ | 29,416 | 1.876 |
| 500 | $(48001 - j8995)$ | 48,836 | $-0.185$ |
| 1600 | $(1657 - j13520)$ | 13,621 | $-1.449$ |
| 1700 | $4723 + j0$ | 4723 | 0 |

**Table 3.1:** Complex amplitudes for harmonic signal that approximates the vowel sound “ah”.


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# Example: Synthetic Vowel

- Sum of 5 Frequency Components

| $k$ | $f_k$ (Hz) | $a_k$ | Mag | Phase |
|---|---|---|---|---|
| 1 | 100 | 0 | 0 | 0 |
| 2 | 200 | $0.00772 + j0.122$ | 0.1223 | 1.508 |
| 3 | 300 | 0 | 0 | 0 |
| 4 | 400 | $-0.08866 + j0.2805$ | 0.2942 | 1.877 |
| 5 | 500 | $0.48 - j0.08996$ | 0.4884 | $-0.185$ |
| 6 | 600 | 0 | 0 | 0 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 15 | 1500 | 0 | 0 | 0 |
| 16 | 1600 | $0.01656 - j0.1352$ | 0.1362 | $-1.449$ |
| 17 | 1700 | $0.04724 + j0$ | 0.04724 | 0 |


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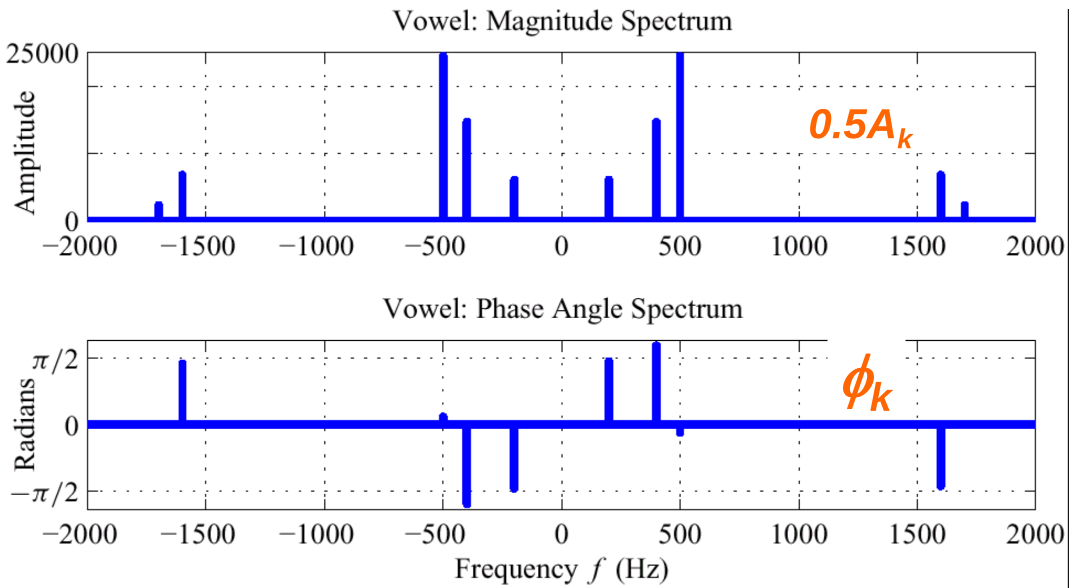
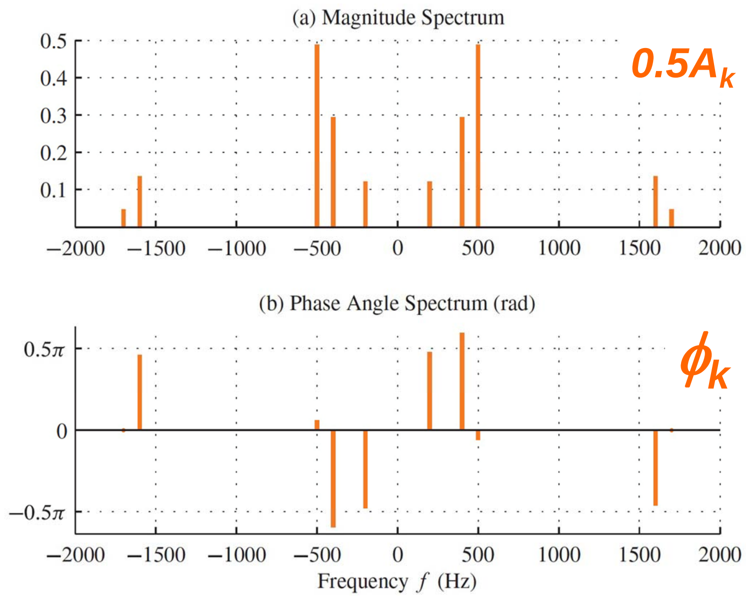
# SPECTRUM of VOWEL (Polar Format)

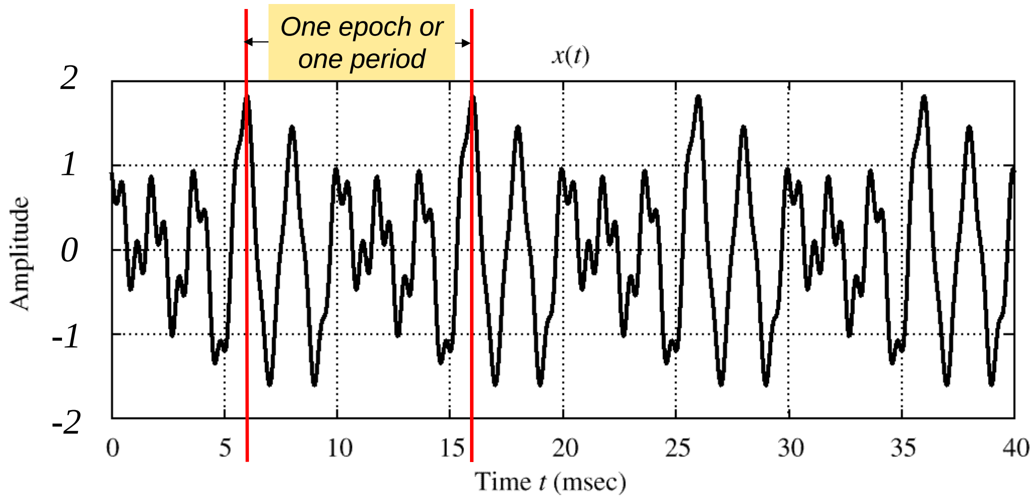


# SPECTRUM of VOWEL (Polar Format)






# Vowel Waveform (sum of all 5 components)




*Note that the period is 10 ms, which equals $1/f_0$*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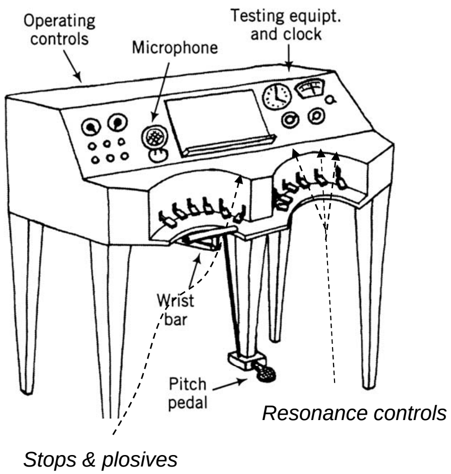
# The VODER by Dudley





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