DSP First, 2/e

Lecture 6

Periodic Signals, Harmonics & Time-Varying Sinusoids

READING ASSIGNMENTS

- This Lecture:
  - Chapter 3, Sections 3-2 and 3-4
  - Chapter 3, Sections 3-6 and 3-7
- Next Lectures:
  - **Fourier Series ANALYSIS**
  - Sections 3-4 and 3-5

LECTURE OBJECTIVES

- Signals with HARMONIC Frequencies
  - Add Sinusoids with $f_k = kf_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

***Second Topic**: FREQUENCY can change vs. TIME*

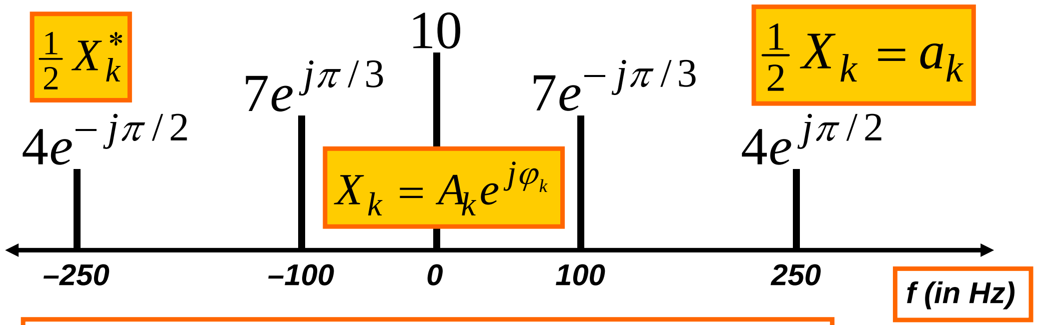
Introduce Spectrogram Visualization

`(spectrogram.m)` **`(plotspec.m)`**

Chirps: $x(t) = \cos(\alpha t^2)$

SPECTRUM DIAGRAM

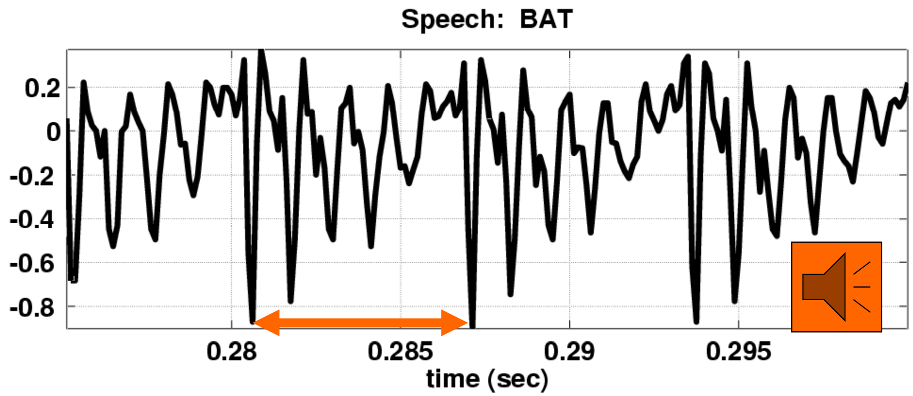
- Recall Complex Amplitude vs. Freq



$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

SPECTRUM for PERIODIC ?

- Nearly Periodic in the Vowel Region
  - Period is (Approximately) T = 0.0065 sec



Harmonic Signal

$$\text{Periodic signal}: x(t) = x(t+T)$$

$$\text{Can only have } \textbf{\textit{harmonic}} \text{ freqs}: f_k = k f_0$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$x(t)$ is periodic if $f_0 T = 1$

$$\cos(2\pi k f_0 (t+T) + \varphi_k) = \cos(2\pi k f_0 t + 2\pi k f_0 T + \varphi_k)$$

Define FUNDAMENTAL FREQ

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_0 = \frac{1}{T_0}$$

Largest f_0 such that

$$f_k = k f_0 \qquad (\omega_0 = 2\pi f_0)$$

f_0 = fundamental Frequency

f_k / f_0 = integer, for all k

T_0 = fundamental Period

Main point: *for periodic signals, all spectral lines have frequencies that are* ***integer*** *multiples of the fundamental frequency*

Harmonic Signal Spectrum

$$\textbf{\textit{Harmonic}} \text{ freqs}: f_k = k f_0$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_0 = \frac{1}{T}$$

$$X_k = A_k e^{j\varphi_k}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t} \right\}$$

Periodic Signal: Example

$$\omega_0 = 2\pi / T$$
$$\Rightarrow \omega_0 T = 2\pi$$

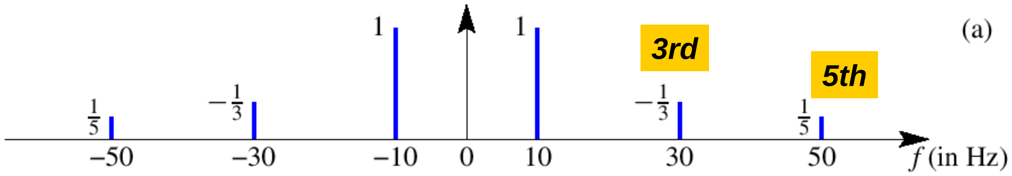
Fundamental frequency

$$e^{j\omega_0(t+T)} = e^{j\omega_0 t} e^{j\omega_0 T} = e^{j\omega_0 t} e^{j2\pi} = e^{j\omega_0 t}$$

$$e^{j7\omega_0(t+T)} = e^{j7\omega_0 t} e^{j14\pi} = e^{j7\omega_0 t}$$

$$x(t+T) = e^{j\omega_0(t+T)} + e^{j7\omega_0(t+T)} + e^{j10\omega_0(t+T)}$$
$$= e^{j\omega_0 t} + e^{j7\omega_0 t} + e^{j10\omega_0 t} = x(t)$$

Harmonic Spectrum (3 Freqs)

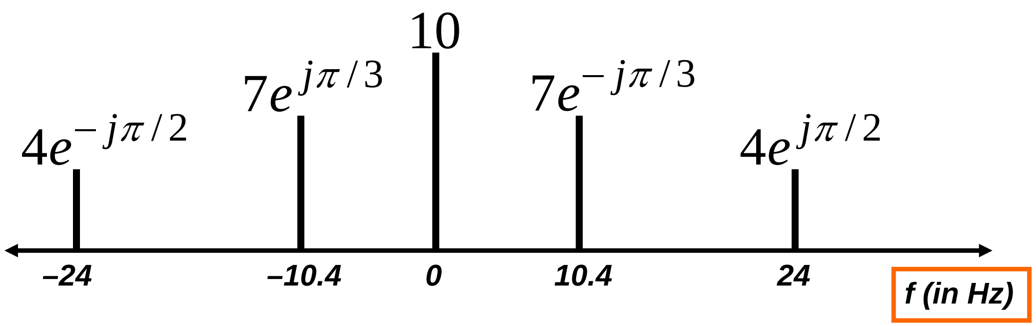


What is the fundamental frequency?

10 Hz

POP QUIZ: FUNDAMENTAL

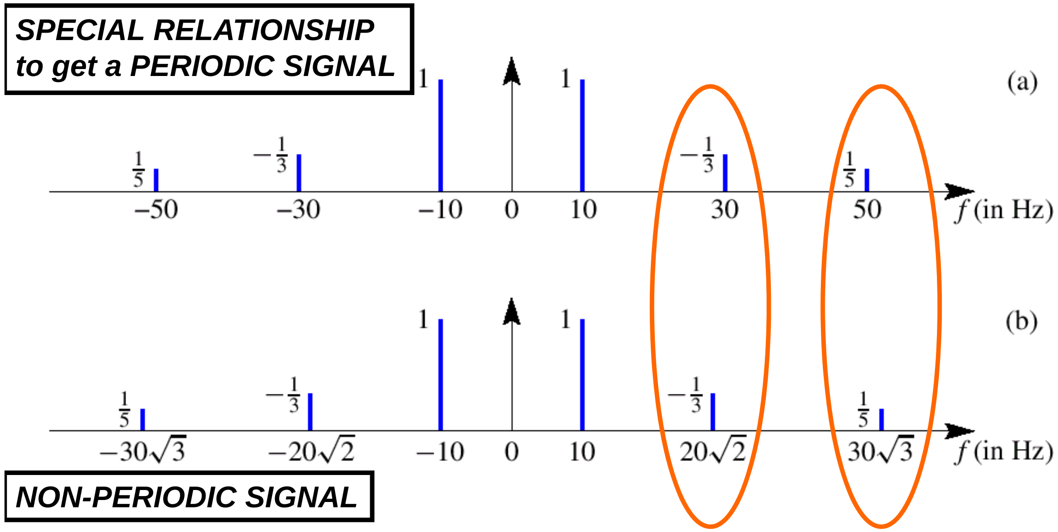
- Here's another spectrum:



What is the fundamental frequency?

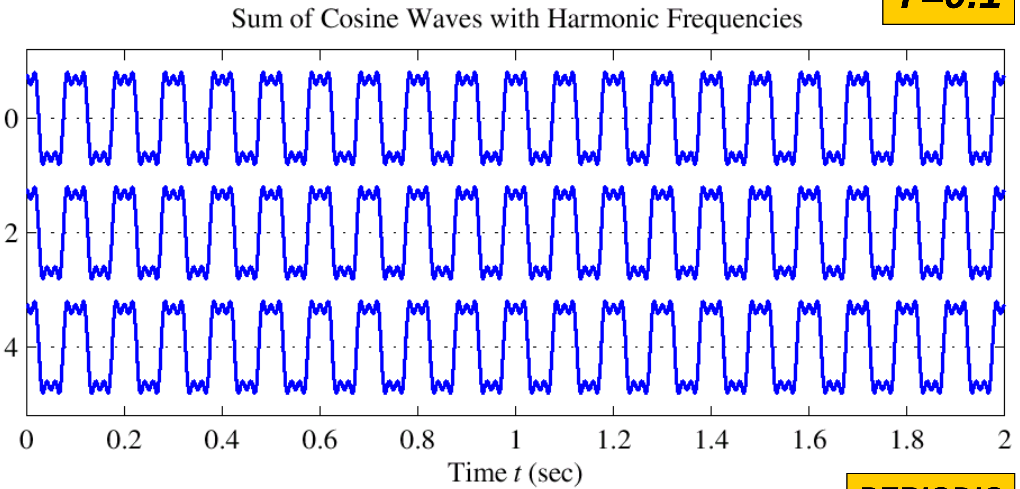
(0.1)GCD(104,240) = (0.1)(8)=0.8 Hz

IRRATIONAL SPECTRUM



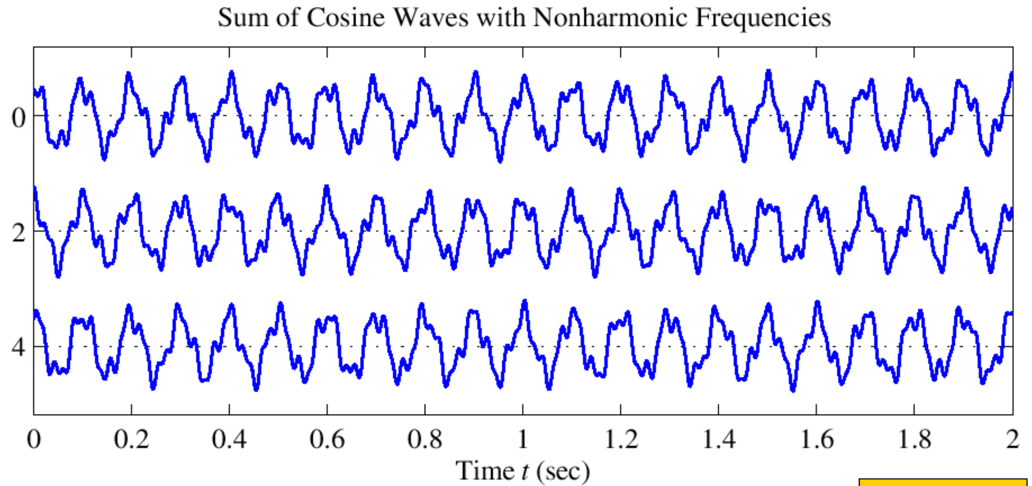
Harmonic Signal (3 Freqs)

T=0.1



PERIODIC

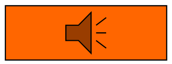
NON-Harmonic Signal

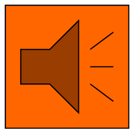


NOT PERIODIC

FREQUENCY ANALYSIS

- **<u>Now, a much HARDER problem</u>**
- Given a recording of a song, have the computer write the music





- *Can a machine extract frequencies?*
  - *Yes, if we COMPUTE the spectrum for x(t)*
    - *During short intervals*

Time-Varying FREQUENCIES Diagram

Frequency is the vertical axis

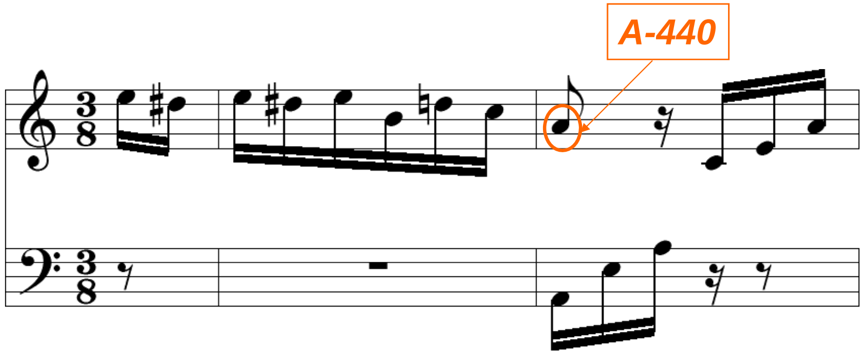
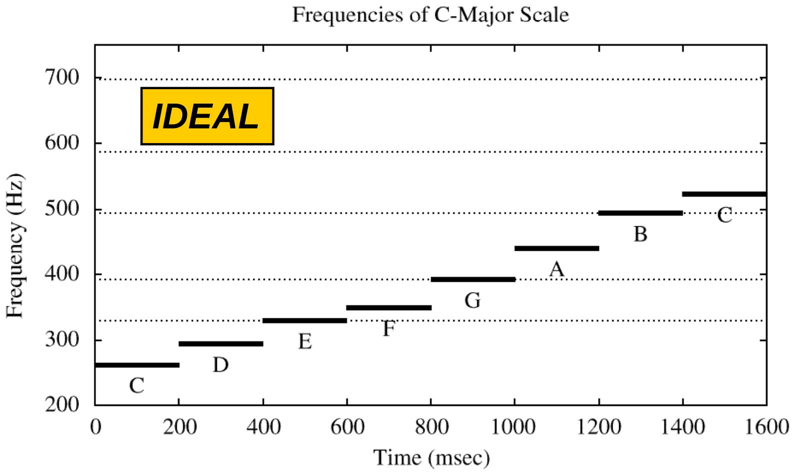


Figure 3.18 Sheet-music notation is a time–frequency diagram.

<u>Time is the horizontal axis</u>

SIMPLE TEST SIGNAL

- C-major SCALE: stepped frequencies
  - Frequency is constant for each note

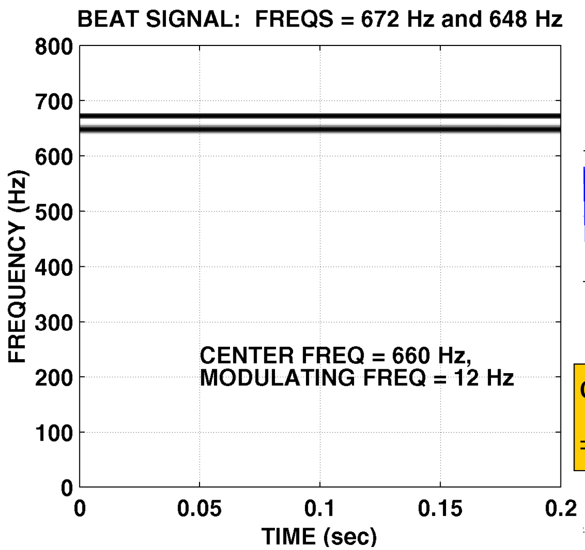


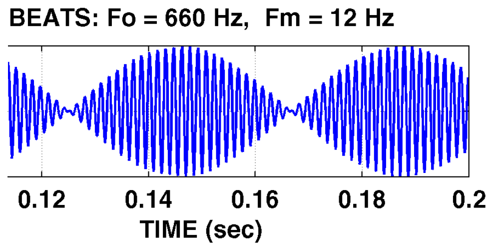
SPECTROGRAM

- SPECTROGRAM Tool
  - MATLAB function is **spectrogram.m**
  - SP-First has **plotspec.m** & **spectgr.m**
- ANALYSIS program
  - Takes x(t) as input
  - Produces spectrum values X_k
  - Breaks x(t) into SHORT TIME SEGMENTS
    - Then uses the FFT (Fast Fourier Transform)

SPECTROGRAM EXAMPLE

- Two **Constant** Frequencies: Beats





$$\cos(2\pi(672)t) + \cos(2\pi(648)t)$$
$$= 2\cos(2\pi(12)t)\cos(2\pi(660)t)$$

AM Radio Signal

- Same form as BEAT Notes, but higher in freq

$$\cos(2\pi(660)t)\sin(2\pi(12)t)$$

$$\frac{1}{2}\left(e^{j2\pi(660)t} + e^{-j2\pi(660)t}\right)\frac{1}{2j}\left(e^{j2\pi(12)t} - e^{-j2\pi(12)t}\right)$$

$$\frac{1}{4j}\left(e^{j2\pi(672)t} - e^{-j2\pi(672)t} - e^{j2\pi(648)t} + e^{-j2\pi(648)t}\right)$$

$$\tfrac{1}{2}\cos(2\pi(672)t - \tfrac{\pi}{2}) + \tfrac{1}{2}\cos(2\pi(648)t + \tfrac{\pi}{2})$$

SPECTRUM of AM (Amplitude Modulation)

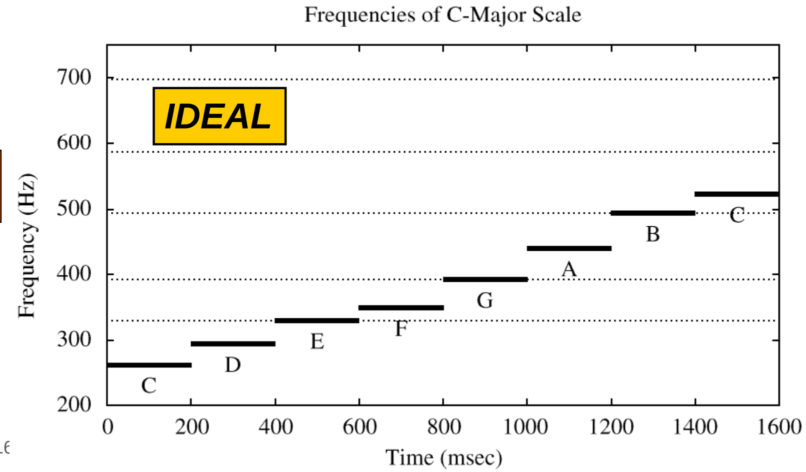
- **SUM** of 4 complex exponentials:

What is the fundamental frequency?

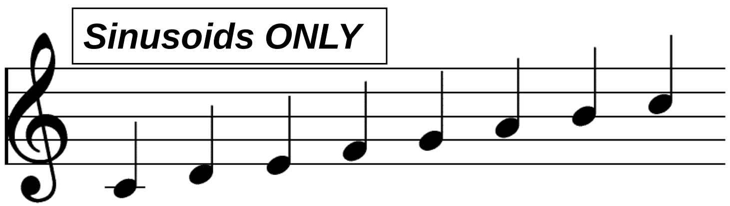
648 Hz ? ***24 Hz ?***

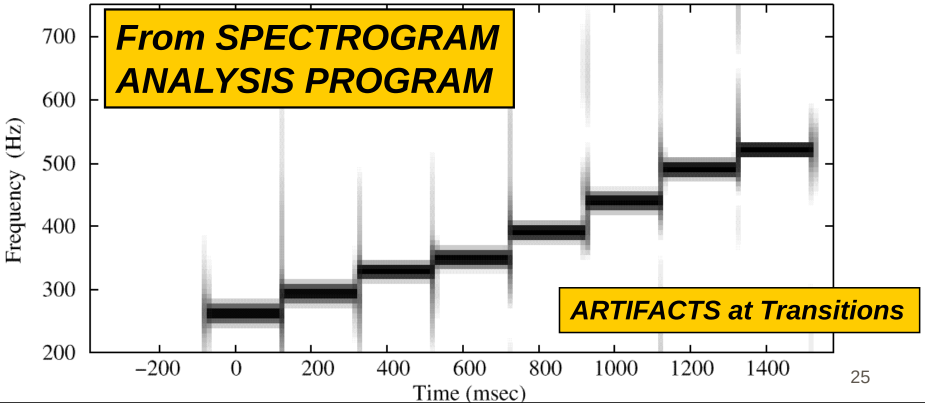
STEPPED FREQUENCIES

- C-major SCALE: successive sinusoids
  - Frequency is constant for each note

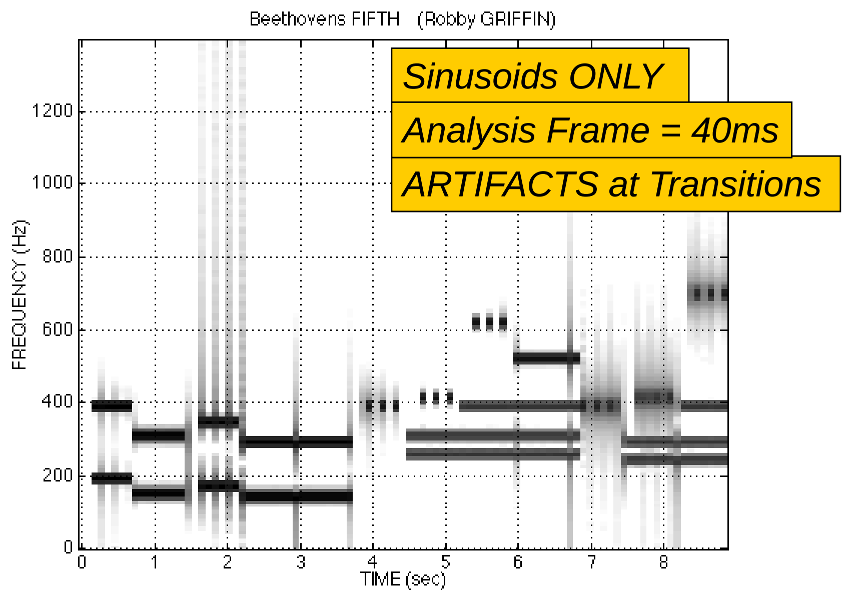


SPECTROGRAM of C-Scale



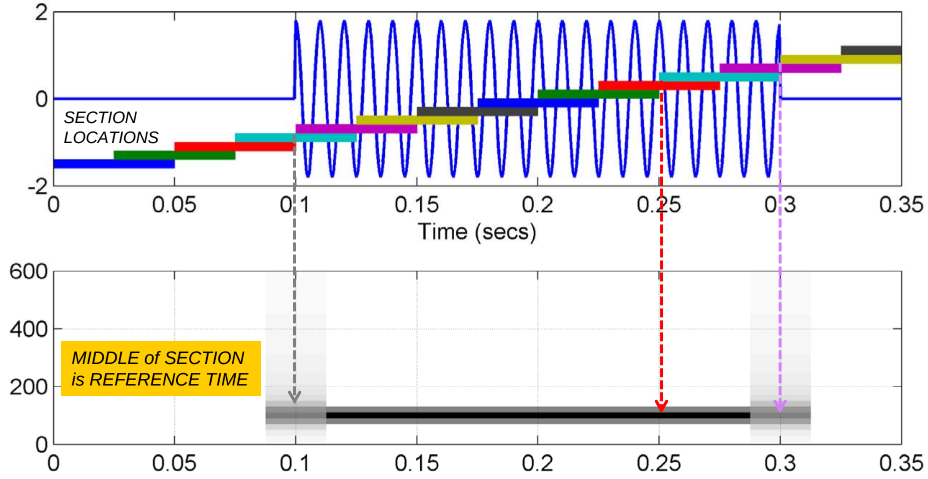


Spectrogram of LAB SONG

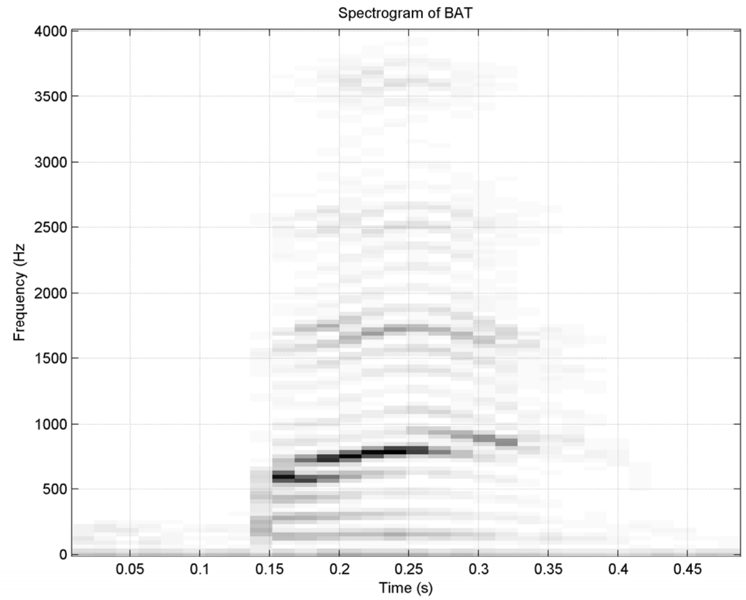


Overlapping Sections in Spectrograms (useful in Labs)

- 50% overlap is common
- Consider edge effects when analyzing a short sinusoid



Spectrogram of BAT (plotspec)



Time-Varying Frequency

- Frequency can change vs. time
  - **Continuously, not stepped**
- **FREQUENCY MODULATION (FM)**

$$x(t) = \cos(2\pi f_c t + v(t))$$

VOICE

- CHIRP SIGNALS
  - Linear Frequency Modulation (LFM)

New Signal: Linear FM

- Called Chirp Signals (LFM)
  - Quadratic phase

QUADRATIC

$$x(t) = A\cos(\alpha t^2 + 2\pi f_0 t + \varphi)$$

- Freq will change LINEARLY vs. time
  - Example of Frequency Modulation (FM)
  - Define "instantaneous frequency"

INSTANTANEOUS FREQ

- Definition

$$x(t) = A\cos(\psi(t))$$
$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t)$$

Derivative of the "Angle"

- For Sinusoid:

$$x(t) = A\cos(2\pi f_0 t + \varphi)$$
$$\psi(t) = 2\pi f_0 t + \varphi$$
$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = 2\pi f_0$$

Makes sense

INSTANTANEOUS FREQ of the Chirp

- Chirp Signals have Quadratic phase
- Freq will change LINEARLY vs. time

$$x(t) = A\cos(\alpha t^2 + \beta t + \varphi)$$
$$\Rightarrow \psi(t) = \alpha t^2 + \beta t + \varphi$$

$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = 2\alpha t + \beta$$

CHIRP SPECTROGRAM