DSP First, 2/e

Lecture 7

Fourier Series Analysis

READING ASSIGNMENTS

- This Lecture:
  - **Fourier Series in Ch 3, Sect. 3-5**
  - Also, periodic signals, Sect. 3-4
- Other Reading:
  - Appendix C: More details on Fourier Series

LECTURE OBJECTIVES

- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}dt$$

- **ANALYSIS** via Fourier Series
  - For **PERIODIC** signals: $\mathbf{x(t+T_0) = x(t)}$
  - Draw spectrum from the Fourier Series coeffs

HISTORY

- Jean Baptiste Joseph Fourier
  - 1807 thesis (memoir)
    - On the Propagation of Heat in Solid Bodies
  - Heat !
  - Napoleonic era
- http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Fourier.html

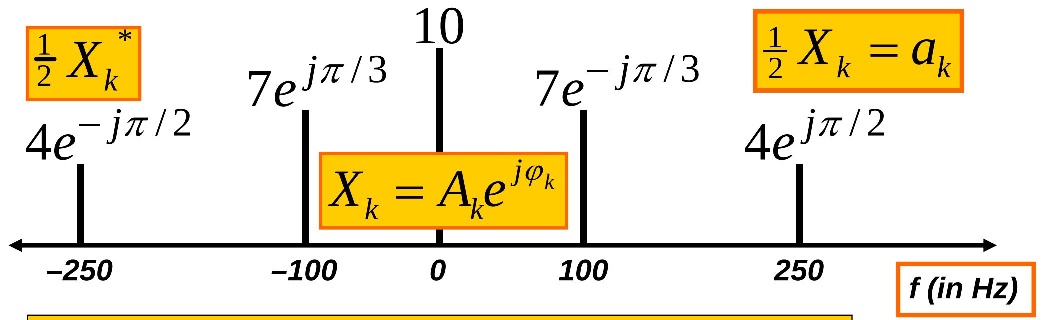
Joseph Fourier

lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^{N} \{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \}$$

Harmonic Signal->Periodic

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k F_0 t}$$

Sums of ***Harmonic*** *complex exponentials are* ***Periodic*** *signals*

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(F_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{F_0}$$

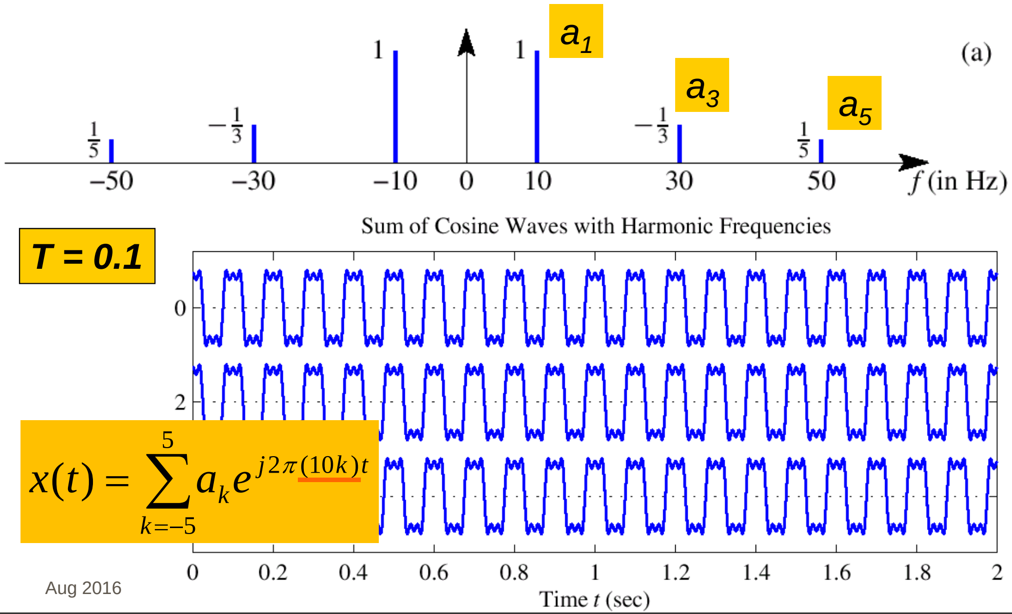
Notation for Fundamental Frequency in Fourier Series

- The k-th frequency is $f_k = kF_0$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k F_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_k t}$$

- Thus, $f_0 = 0$ is DC
- This is why we use upper case F_0 for the Fundamental Frequency

Harmonic Signal (3 Freqs)



Periodic signals->Harmonic?

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k F_0 t}$$

Can all ***periodic*** *signals be written as* ***harmonic*** *signals?*

- Fourier's contribution was to postulate the answer is yes
  - Called **Fourier Series**
- For heat transfer it is easy to solve PDE for sinusoidal sources, but difficult for general sources
- Made formal by Dirichlet and Riemann

STRATEGY: x(t) → a_k

- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
  - Measure similarity between signal & harmonic
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

CALCULUS for complex exp

$$\frac{d}{dt} e^{\alpha t} = \alpha e^{\alpha t} \quad \rightarrow \quad \frac{d}{dt} e^{j\alpha t} = j\alpha e^{\alpha t}$$

$$\int_a^b e^{\beta t} dt = \frac{1}{\beta} e^{\beta t}\Big|_a^b = \frac{1}{\beta}\left(e^{\beta b} - e^{\beta a}\right)$$

$$\int_a^b e^{j\beta t} dt = \frac{1}{j\beta} e^{j\beta t}\Big|_a^b = \frac{1}{j\beta}\left(e^{j\beta b} - e^{j\beta a}\right)$$

INTEGRAL Property of exp(j)

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}dt = \frac{T_0}{-j2\pi m} e^{-j(2\pi/T_0)mt}\Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m}(e^{-j2\pi m} - 1)$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}dt = 0 \quad m \neq 0$$

$$\omega_0 = \frac{2\pi}{T_0}$$

ORTHOGONALITY of exp(j)

$$\frac{1}{T_0}\int_0^{T_0} \underbrace{e^{j(2\pi/T_0)\ell t} e^{-j(2\pi/T_0)kt}}_{e^{j(2\pi/T_0)(\ell-k)t}} dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

$$m = \ell - k$$

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)(\ell-k)t}dt$$

Fourier Series Integral

- Use orthogonality to determine a_k from x(t)

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

Fundamental Freq.
$F_0 = 1/T_0$

$$a_{-k} = a_k^* \quad \text{when } x(t) \text{ is real}$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt \qquad \text{(DC component)}$$

Isolate One FS Coefficient

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

$$\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)\ell t}dt = \frac{1}{T_0}\int_0^{T_0}\left(\sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}\right)e^{-j(2\pi/T_0)\ell t}dt$$

$$\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)\ell t}dt = \sum_{k=-\infty}^{\infty} a_k \left(\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)kt}e^{-j(2\pi/T_0)\ell t}dt\right) = a_\ell$$

Integral is zero except for $k = \ell$

$$\Rightarrow a_\ell = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)\ell t}dt$$

ℓ is dummy variable, could be k

Fourier Series: x(t) → a_k

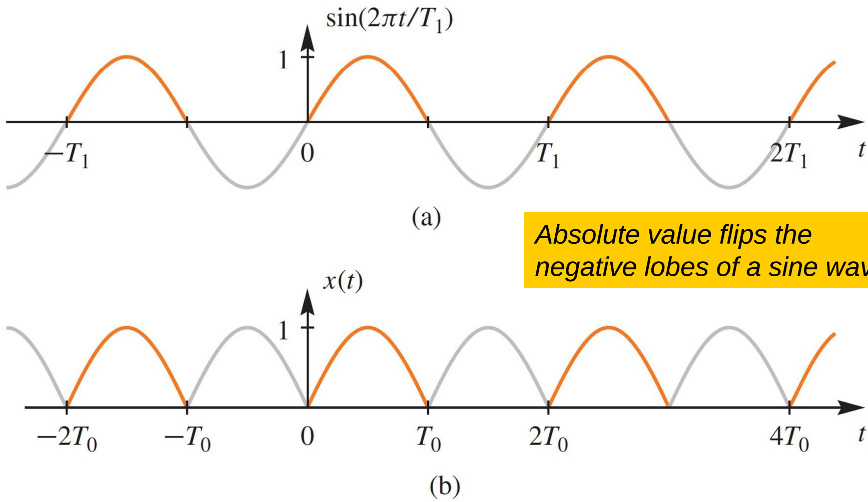
- **ANALYSIS**
  - Given a **PERIODIC Signal**
  - Fourier Series coefficients are obtained via an **INTEGRAL over one period**

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}dt$$

  - Next, consider a specific signal, the FWRS
    - Full Wave Rectified Sine

Full-Wave Rectified Sine

$x(t) = |\sin(2\pi t/T_1)|$ Period is $T_0 = \frac{1}{2}T_1$



Absolute value flips the negative lobes of a sine wave

Full-Wave Rectified Sine {a_k}

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

Full-Wave Rectified Sine

$x(t) = |\sin(2\pi t/T_1)|$

Period : $T_0 = \frac{1}{2}T_1$

$\Rightarrow x(t) = |\sin(\pi t/T_0)|$

$$a_k = \frac{1}{T_0}\int_0^{T_0}\sin(\tfrac{\pi}{T_0}t)e^{-j(2\pi/T_0)kt}dt$$

$$= \frac{1}{T_0}\int_0^{T_0}\frac{e^{j(\pi/T_0)t} - e^{-j(\pi/T_0)t}}{2j}e^{-j(2\pi/T_0)kt}dt$$

$$= \frac{1}{j2T_0}\int_0^{T_0} e^{-j(\pi/T_0)(2k-1)t}dt - \frac{1}{j2T_0}\int_0^{T_0} e^{-j(\pi/T_0)(2k+1)t}dt$$

$$= \frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))}\Bigg|_0^{T_0} - \frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))}\Bigg|_0^{T_0}$$

Full-Wave Rectified Sine {a_k}

$$a_k = \frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))}\Bigg|_0^{T_0} - \frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))}\Bigg|_0^{T_0}$$

$$= \frac{1}{2\pi(2k-1)}\left(e^{-j(\pi/T_0)(2k-1)T_0} - 1\right) - \frac{1}{2\pi(2k+1)}\left(e^{-j(\pi/T_0)(2k+1)T_0} - 1\right)$$

$$= \frac{1}{\pi(2k-1)}\left(e^{-j\pi(2k-1)} - 1\right) - \frac{1}{\pi(2k+1)}\left(e^{-j\pi(2k+1)} - 1\right)$$

$$= \left(\frac{2k+1-(2k-1)}{\pi(4k^2-1)}\right)\left(-(-1)^{2k} - 1\right) = \frac{-2}{\pi(4k^2-1)}$$

Fourier Coefficients: a_k

- a_k is a function of k
  - Complex Amplitude for k-th Harmonic

$$a_k = \frac{-2}{\pi(4k^2-1)}$$

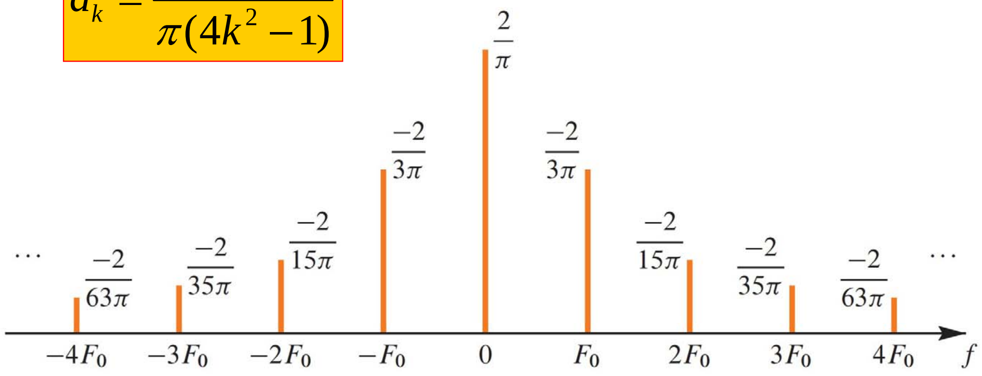
  - Does not depend on the period, T_0
  - DC value is $a_0 = 2/\pi = 0.6336$

Spectrum from Fourier Series

Plot a_k for Full-Wave Rectified Sinusoid

$$a_k = \frac{-2}{\pi(4k^2-1)}$$

$F_0 = 1/T_0$ and $\omega_0 = 2\pi F_0$



Reconstruct From Finite Number of Harmonic Components

Full-Wave Rectified Sinusoid $x(t) = |\sin(\pi t / T_0)|$

$T_0 = 10\,\text{ms} \Rightarrow F_0 = 100\,\text{Hz}$

$$a_k = \frac{-2}{\pi(4k^2-1)}$$

$a_0 = 2/\pi = 0.6336$

$$x_N(t) = a_0 + \sum_{k=1}^{N} \left\{ a_k e^{j2\pi k F_0 t} + a_k^* e^{-j2\pi k F_0 t} \right\}$$

How close is $x_N(t)$ to $x(t) = |\sin(\pi t / T_0)|$?

Reconstruct From Finite Number of Spectrum Components

Full-Wave Rectified Sinusoid $x(t) = |\sin(\pi t / T_0)|$

$T_0 = 10\,\text{ms} \Rightarrow F_0 = 100\,\text{Hz}$

$a_0 = 2/\pi = 0.6336$

$$a_k = \frac{-2}{\pi(4k^2-1)}$$