DSP First, 2/e

Lecture 7C

Fourier Series Examples:

Common Periodic Signals

READING ASSIGNMENTS

- This Lecture:
  - **Appendix C, Section C-2**
  - **Various Fourier Series**
    - **Pulse Waves**
    - **Triangular Wave**
    - **Rectified Sinusoids (also in Ch. 3, Sect. 3-5)**

LECTURE OBJECTIVES

- Use the Fourier Series Integral

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}dt$$

- Derive Fourier Series coeffs for common periodic signals
- Draw spectrum from the Fourier Series coeffs
  - a_k is Complex Amplitude for k-th Harmonic

Harmonic Signal is Periodic

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k F_0 t}$$

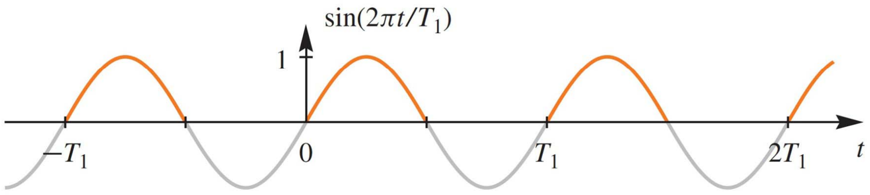
*Sums of **Harmonic** complex exponentials are **Periodic** signals*

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(F_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{F_0}$$

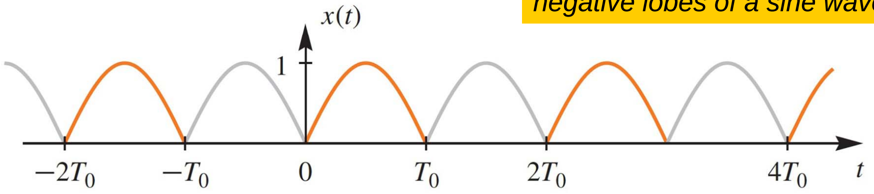
Recall FWRS

$x(t) = |\sin(2\pi t / T_1)|$ Period is $T_0 = \frac{1}{2} T_1$



(a)

Absolute value flips the negative lobes of a sine wave



(b)

FWRS Fourier Integral → {a_k}

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt}\,dt$$

Full-Wave Rectified Sine

$x(t) = |\sin(2\pi t / T_1)|$

$Period: T_0 = \frac{1}{2} T_1$

$\Rightarrow x(t) = |\sin(\pi t / T_0)|$

$$a_k = \tfrac{1}{T_0}\int_0^{T_0} \sin(\tfrac{\pi}{T_0} t) e^{-j(2\pi/T_0)kt}\,dt$$

$$= \tfrac{1}{T_0}\int_0^{T_0} \frac{e^{j(\pi/T_0)t} - e^{-j(\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt}\,dt$$

$$= \left.\frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))}\right|_0^{T_0} - \left.\frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))}\right|_0^{T_0}$$

$$a_k = \frac{-2}{\pi(4k^2-1)}$$

FWRS Fourier Coeffs: a_k

- a_k is a function of k
  - Complex Amplitude for k-th Harmonic

$$a_k = \frac{-2}{\pi(4k^2-1)}$$

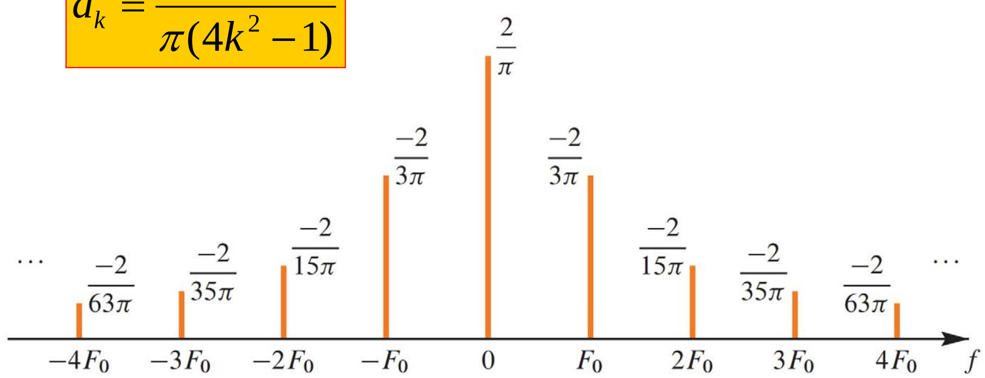
  - Does not depend on the period, T_0
  - DC value is $a_0 = 2/\pi = 0.6336$

Spectrum from Fourier Series

Plot a_k for Full-Wave Rectified Sinusoid

$$a_k = \frac{-2}{\pi(4k^2-1)}$$

$F_0 = 1/T_0$ and $\omega_0 = 2\pi F_0$



Fourier Series Synthesis

- HOW do you **APPROXIMATE** x(t) ?

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

- Use FINITE number of coefficients

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k F_0 t}$$

$a_{-k} = a_k^*$ when $x(t)$ is real

Reconstruct From Finite Number of Harmonic Components

Full-Wave Rectified Sinusoid $x(t) = |\sin(\pi t / T_0)|$

$T_0 = 10\,\text{ms}$
$\Rightarrow F_0 = 100\,\text{Hz}$

$$a_k = \frac{-2}{\pi(4k^2-1)}$$

$a_0 = 2/\pi = 0.6336$

$$x_N(t) = a_0 + \sum_{k=1}^{N}\left\{a_k e^{j2\pi k F_0 t} + a_k^* e^{-j2\pi k F_0 t}\right\}$$

How close is $x_N(t)$ to $x(t) = |\sin(\pi t / T_0)|$?

Full-Wave Rectified Sine $\{a_k\}$

$$a_k = \frac{-2}{\pi(4k^2-1)} = \frac{-2}{\pi(4k^2-1)} \quad \text{is real-valued}$$

$$x_N(t) = \sum_{k=-N}^{N} \frac{-2}{\pi(4k^2-1)} e^{jk\omega_0 t}$$

$$= \frac{-2}{-\pi} + \frac{-2}{\pi(4-1)}e^{j\omega_0 t} + \frac{-2}{\pi(4-1)}e^{-j\omega_0 t} + \frac{-2}{\pi(16-1)}e^{j2\omega_0 t} + \frac{-2}{\pi(16-1)}e^{-j2\omega_0 t}\ldots$$

$$= \frac{2}{\pi} - \frac{2}{3\pi}e^{j\omega_0 t} - \frac{2}{3\pi}e^{-j\omega_0 t} - \frac{2}{15\pi}e^{j2\omega_0 t} - \frac{2}{15\pi}e^{-j2\omega_0 t} + \ldots$$

$$= \frac{2}{\pi} - \frac{4}{3\pi}\cos(\omega_0 t) - \frac{4}{15\pi}\cos(2\omega_0 t) - \ldots - \frac{4}{(4N^2-1)\pi}\cos(N\omega_0 t)$$

- Plots for N=4 and N=9 are shown next
- Excellent Approximation for N=9

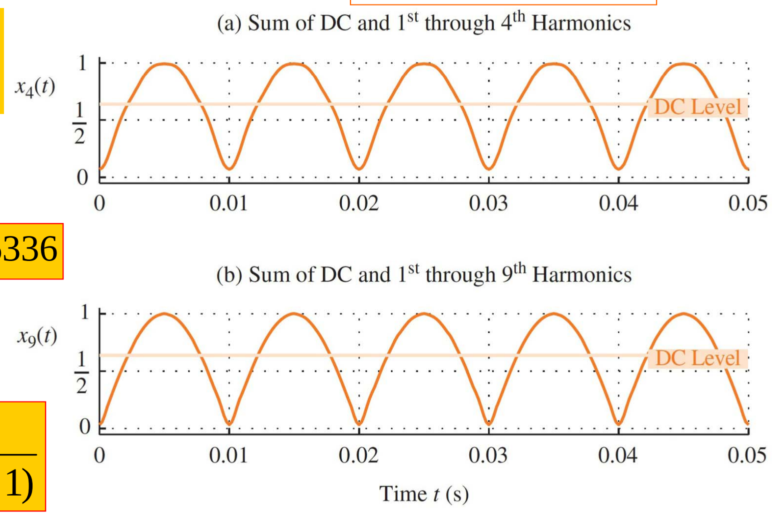
Reconstruct From Finite Number of Spectrum Components

Full-Wave Rectified Sinusoid $x(t) = |\sin(\pi t / T_0)|$

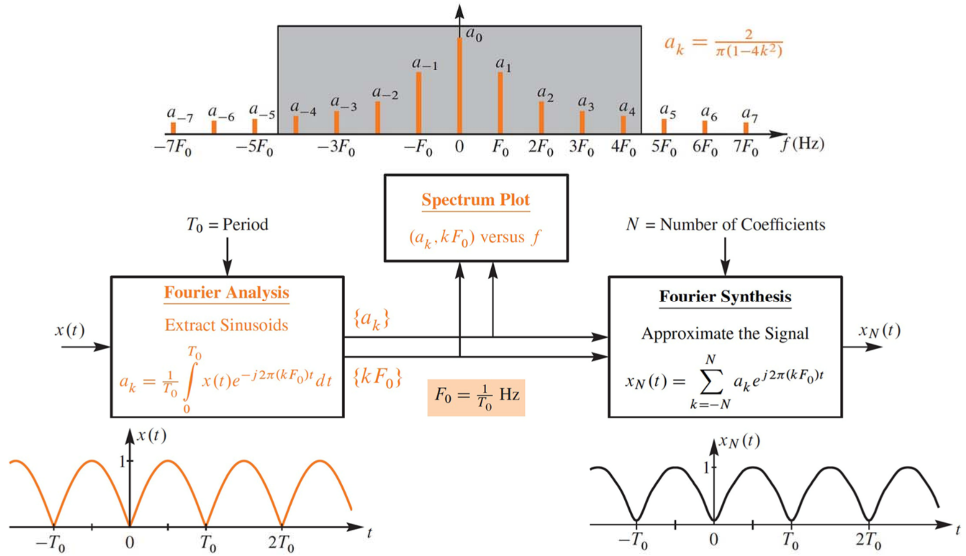
$T_0 = 10\,\text{ms}$
$\Rightarrow F_0 = 100\,\text{Hz}$

$a_0 = 2/\pi = 0.6336$

$$a_k = \frac{-2}{\pi(4k^2-1)}$$



Fourier Series Synthesis

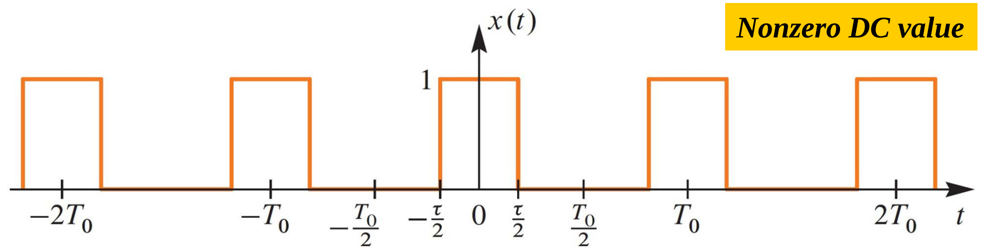


PULSE WAVE SIGNAL GENERAL FORM

Defined over one period

$$x(t) = \begin{cases} 1 & 0 \le |t| < \tau/2 \\ 0 & \tau/2 \le |t| \le T_0/2 \end{cases}$$

Nonzero DC value



Pulse Wave $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j(2\pi/T_0)(k)t}dt$$

General PulseWave

$$x(t) = \begin{cases} 1 & 0 \le |t| < \tau/2 \\ 0 & \tau/2 \le |t| \le T_0/2 \end{cases}$$

$$a_k = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} 1e^{-j(2\pi/T_0)kt}dt$$

$$= (\tfrac{1}{T_0})\frac{e^{-j(2\pi/T_0)kt}}{-j(2\pi/T_0)k}\Bigg|_{-\tau/2}^{\tau/2} = \frac{e^{-j(2\pi/T_0)k(\tau/2)} - e^{-j(2\pi/T_0)k(-\tau/2)}}{-j(2\pi)k}$$

$$= \frac{e^{j(\pi/T_0)k(\tau)} - e^{-j(\pi/T_0)k(\tau)}}{(j2)\pi k} = \frac{\sin(\pi k\tau/T_0)}{\pi k}$$

Pulse Wave $\{a_k\}$ = sinc

PulseWave

$$a_k = \frac{\sin(\pi k\tau/T_0)}{\pi k} \qquad k = 0, \pm1, \pm2, \ldots$$

Double check the DC coefficient:

$$a_0 = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} 1e^{-j(2\pi/T_0)(0)t}dt$$

$$= \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} 1dt = \frac{1}{T_0}\left[\frac{\tau}{2} - \frac{-\tau}{2}\right] = \frac{\tau}{T_0}$$

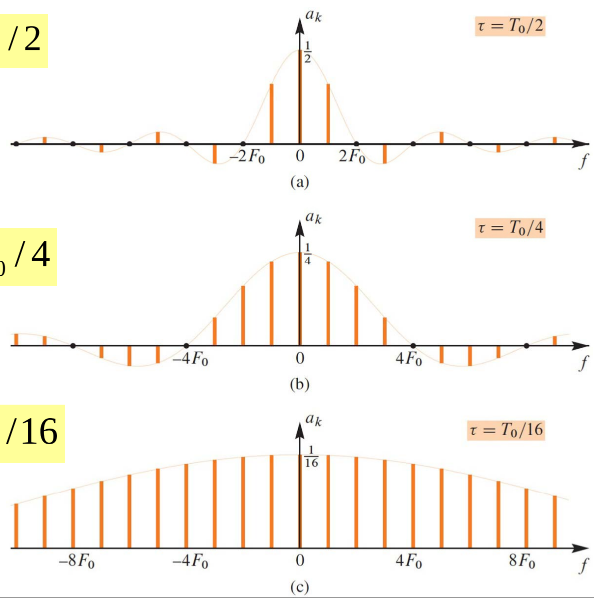
Note, $\lim_{k\to 0} \frac{\sin(\pi k\tau/T_0)}{\pi k} \to \frac{\tau}{T_0}$

PULSE WAVE SPECTRA

$\tau = T_0/2$

$\tau = T_0/4$

$\tau = T_0/16$



50% duty-cycle (Square) Wave

$$\tau = T_0/2 \Rightarrow a_k = \frac{\sin(\pi k (T_0/2)/T_0)}{\pi k} = \frac{\sin(\pi k/2)}{\pi k} \quad k = 0, \pm 1, \pm 2, \ldots$$

- *Thus, a_k=0 when k is odd*
  - *Phase is zero because x(t) is centered at t=0*
    - *different from a previous case*

PulseWave starting at t=0

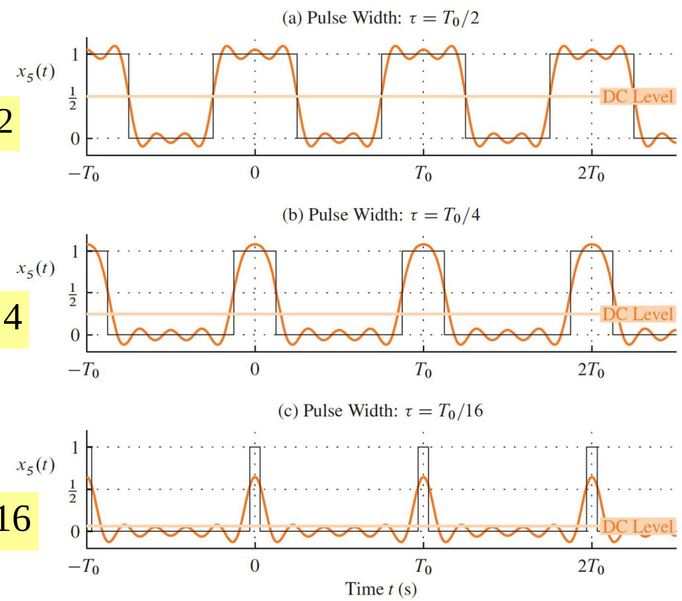
$$x(t) = \begin{cases} 1 & 0 \le |t| < \tau \\ 0 & \tau \le |t| \le T_0 \end{cases} \leftrightarrow a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

PULSE WAVE SYNTHESIS with first 5 Harmonics

$\tau = T_0/2$

$\tau = T_0/4$

$\tau = T_0/16$

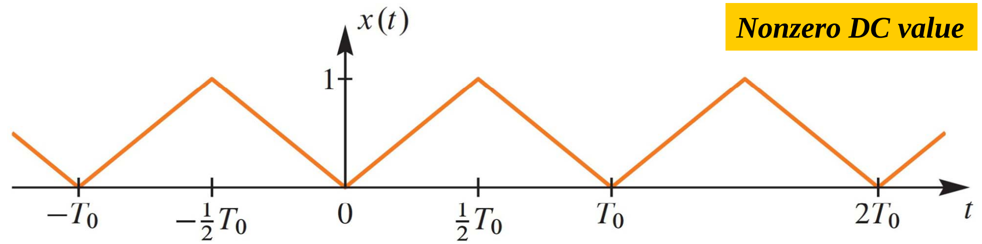


Triangular Wave: Time Domain

Defined over one period

$$x(t) = |2t/T_0| \quad \text{for } -T_0/2 < t \le T_0/2$$

Nonzero DC value



Triangular Wave $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j(2\pi/T_0)kt}dt$$

Triangular Wave

$$x(t) = |2t/T_0|$$

$$\omega_0 = 2\pi/T_0$$

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |2t/T_0|\, e^{-j(2\pi/T_0)kt}dt$$

$$= \frac{2}{T_0^2}\int_0^{T_0/2} te^{-j\omega_0 kt}dt + \frac{2}{T_0^2}\int_{-T_0/2}^{0}(-t)e^{-j\omega_0 kt}dt$$

$$a_k = \frac{2}{T_0^2}e^{-j\omega_0 kt}\left(\frac{j\omega_0 kt+1}{\omega_0^2k^2}\right)\Bigg|_0^{T_0/2} - \frac{2}{T_0^2}e^{-j\omega_0 kt}\left(\frac{j\omega_0 kt+1}{\omega_0^2k^2}\right)\Bigg|_{-T_0/2}^{0}$$

$$= \frac{2}{T_0^2}\left(e^{-j\omega_0 kT_0/2}\left(\frac{j\omega_0 kT_0/2+1}{\omega_0^2k^2}\right) - \frac{1}{\omega_0^2k^2}\right) - \frac{2}{T_0^2}\left(\frac{1}{\omega_0^2k^2} - e^{j\omega_0 kT_0/2}\left(\frac{-j\omega_0 kT_0/2+1}{\omega_0^2k^2}\right)\right)$$

use the indefinite integral $\int te^{-j\omega_0 kt}dt = e^{-j\omega_0 kt}\left(\frac{j\omega_0 kt+1}{\omega_0^2k^2}\right)$

Triangular Wave $\{a_k\}$

$$a_k = \frac{2}{T_0^2}\left[e^{-j\omega_0 kT_0/2}\left(\frac{j\omega_0 kT_0/2+1}{\omega_0^2k^2}\right) - \frac{1}{\omega_0^2k^2}\right] - \frac{2}{T_0^2}\left[\frac{1}{\omega_0^2k^2} - e^{j\omega_0 kT_0/2}\left(\frac{-j\omega_0 kT_0/2+1}{\omega_0^2k^2}\right)\right]$$

$$= \frac{-4}{T_0^2\omega_0^2k^2} + \frac{2}{T_0^2}e^{-j\pi k}\left(\frac{j\pi k+1}{\omega_0^2k^2}\right) + \frac{2}{T_0^2}e^{j\pi k}\left(\frac{-j\pi k+1}{\omega_0^2k^2}\right)$$

$$\omega_0 T_0 = 2\pi \;\Rightarrow\; \frac{4}{T_0^2\omega_0^2} = \frac{1}{\pi^2}$$

$$= \frac{-1}{\pi^2k^2} + \frac{2(-1)^k}{T_0^2}\left(\frac{j\pi k+1}{T_0^2\omega_0^2k^2} + \frac{-j\pi k+1}{T_0^2\omega_0^2k^2}\right) = \frac{-1}{\pi^2k^2} + \left(\frac{4(-1)^k}{T_0^2\omega_0^2k^2}\right)$$

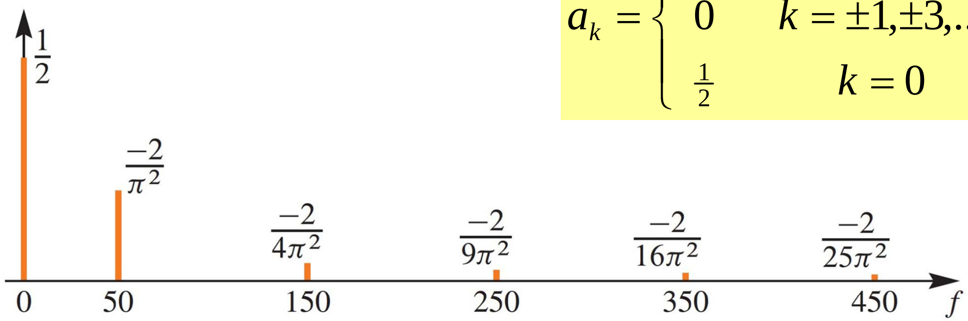
$$= \frac{-1}{\pi^2k^2} + \left(\frac{(-1)^k}{\pi^2k^2}\right) = \begin{cases} \frac{-1}{\pi^2k^2} & k = \pm 2, \pm 4, \ldots \\ 0 & k = \pm 1, \pm 3, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

$$\text{DC} = \frac{\text{Area}}{\text{Period}} \;\Rightarrow\; a_0 = \frac{T_0/2}{T_0} = \frac{1}{2}$$

Triangular Wave $\{a_k\}$

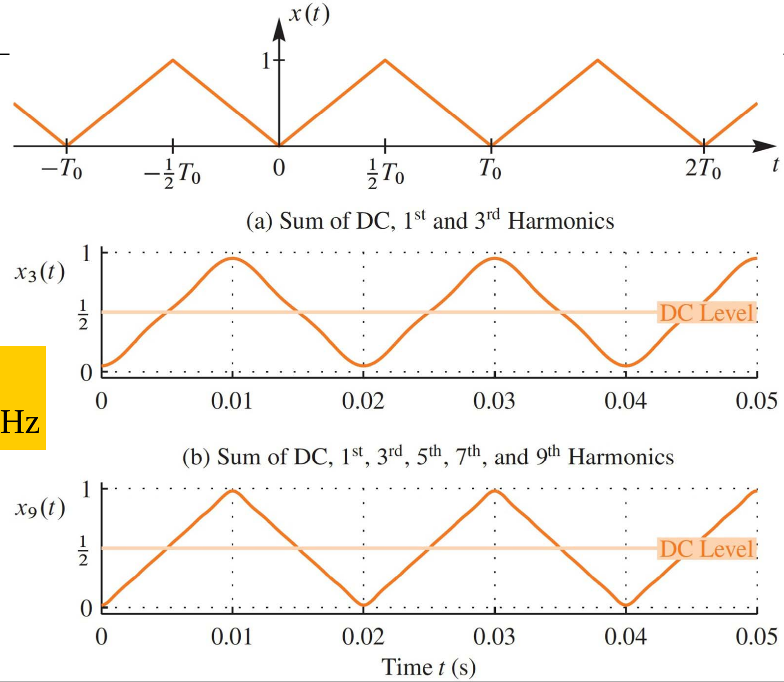
- Spectrum, assuming 50 Hz is the fundamental frequency

$$a_k = \begin{cases} \frac{-1}{\pi^2k^2} & k = \pm 2, \pm 4, \ldots \\ 0 & k = \pm 1, \pm 3, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$



Triangular Wave Synthesis

$T_0 = 20\,\text{ms} \Rightarrow F_0 = 50\,\text{Hz}$



Full-Wave Rectified Sine $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

$$a_k = \tfrac{1}{T_0}\int_0^{T_0} \sin(\tfrac{\pi}{T_0}t)e^{-j(2\pi/T_0)kt}dt$$

$$= \tfrac{1}{T_0}\int_0^{T_0} \frac{e^{j(\pi/T_0)t} - e^{-j(\pi/T_0)t}}{2j}e^{-j(2\pi/T_0)kt}dt$$

$$= \tfrac{1}{j2T_0}\int_0^{T_0} e^{-j(\pi/T_0)(2k-1)t}dt - \tfrac{1}{j2T_0}\int_0^{T_0} e^{-j(\pi/T_0)(2k+1)t}dt$$

$$= \left.\frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))}\right|_0^{T_0} - \left.\frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))}\right|_0^{T_0}$$

Full-Wave Rectified Sine

$$x(t) = |\sin(2\pi t/T_1)|$$

$$Period: T_0 = \tfrac{1}{2}T_1$$

Full-Wave Rectified Sine $\{a_k\}$

$$a_k = \left.\frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))}\right|_0^{T_0} - \left.\frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))}\right|_0^{T_0}$$

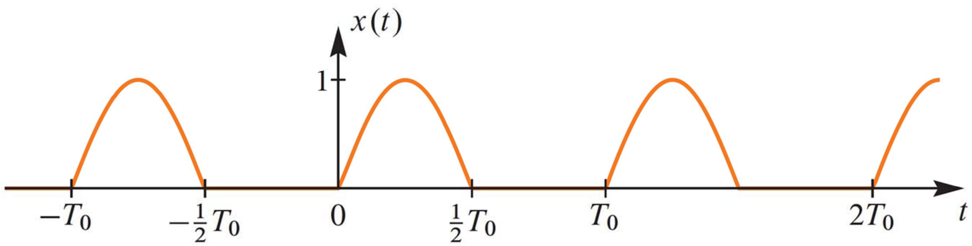
$$= \tfrac{1}{2\pi(2k-1)}\left(e^{-j(\pi/T_0)(2k-1)T_0} - 1\right) - \tfrac{1}{2\pi(2k+1)}\left(e^{-j(\pi/T_0)(2k+1)T_0} - 1\right)$$

$$= \tfrac{1}{\pi(2k-1)}\left(e^{-j\pi(2k-1)} - 1\right) - \tfrac{1}{\pi(2k+1)}\left(e^{-j\pi(2k+1)} - 1\right)$$

$$= \left(\tfrac{2k+1-(2k-1)}{\pi(4k^2-1)}\right)\left(-(-1)^{2k} - 1\right) = \frac{-2}{\pi(4k^2-1)}$$

Half-Wave Rectified Sine

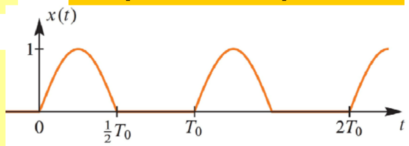
- Signal is positive half cycles of sine wave
- HWRS = Half-Wave Recitified Sine



Half-Wave Rectified Sine $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \quad (k \neq \pm 1)$$

Half-Wave Rectified Sine



$$a_k = \tfrac{1}{T_0}\int_0^{T_0/2} \sin(\tfrac{2\pi}{T_0}t)e^{-j(2\pi/T_0)kt}dt$$

$$= \tfrac{1}{T_0}\int_0^{T_0/2} \frac{e^{j(2\pi/T_0)t} - e^{-j(2\pi/T_0)t}}{2j}e^{-j(2\pi/T_0)kt}dt$$

$$= \tfrac{1}{j2T_0}\int_0^{T_0/2} e^{-j(2\pi/T_0)(k-1)t}dt - \tfrac{1}{j2T_0}\int_0^{T_0/2} e^{-j(2\pi/T_0)(k+1)t}dt$$

$$= \left.\frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))}\right|_0^{T_0/2} - \left.\frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))}\right|_0^{T_0/2}$$

Half-Wave Rectified Sine $\{a_k\}$

$$a_k = \left.\frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))}\right|_0^{T_0/2} - \left.\frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))}\right|_0^{T_0/2}$$

$$= \frac{1}{4\pi(k-1)}\left(e^{-j(2\pi/T_0)(k-1)T_0/2} - 1\right) - \frac{1}{4\pi(k+1)}\left(e^{-j(2\pi/T_0)(k+1)T_0/2} - 1\right)$$

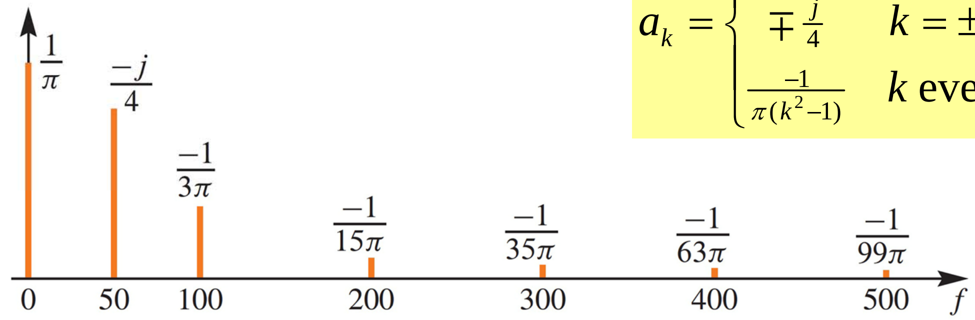
$$= \frac{1}{4\pi(k-1)}\left(e^{-j\pi(k-1)} - 1\right) - \frac{1}{4\pi(k+1)}\left(e^{-j\pi(k+1)} - 1\right)$$

$$= \left(\frac{k+1-(k-1)}{4\pi(k^2-1)}\right)\left(-(-1)^k - 1\right) = \begin{cases} 0 & k \text{ odd} \\ \pm\frac{1}{j4} & k = \pm 1 \\ \frac{-1}{\pi(k^2-1)} & k \text{ even} \end{cases}$$

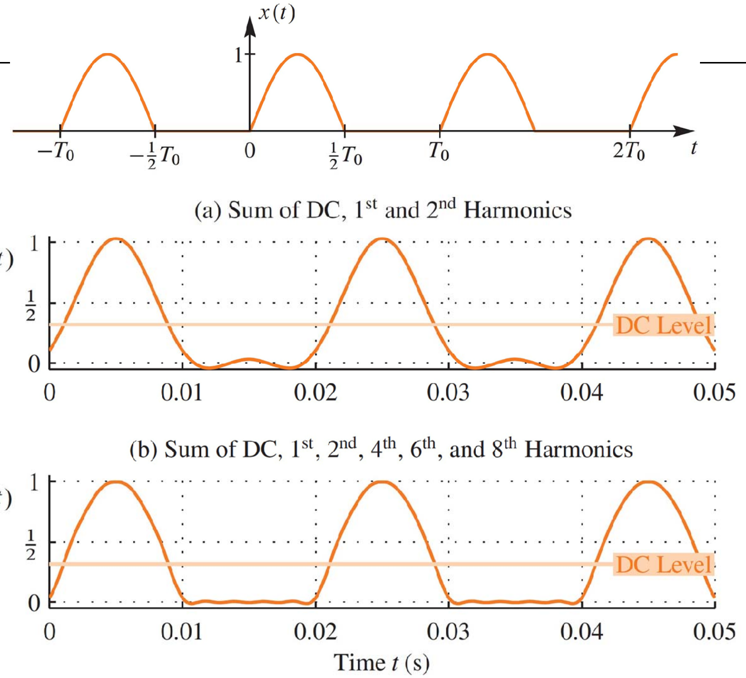
Half-Wave Rectified Sine $\{a_k\}$

- Spectrum, assuming 50 Hz is the fundamental frequency

$$a_k = \begin{cases} 0 & k \text{ odd} \\ \mp\frac{j}{4} & k = \pm 1 \\ \frac{-1}{\pi(k^2-1)} & k \text{ even} \end{cases}$$



HWRS Synthesis



$T_0 = 20\,\text{ms}$

$\Rightarrow F_0 = 50\,\text{Hz}$

Fourier Series Demo

- MATLAB GUI: fseriesdemo
  - Shows the convergence with more terms
  - One of the demos in:
  - http://dspfirst.gatech.edu/matlab/

fseriesdemo GUI