# DSP First, 2/e

## Lecture 8
## Fourier Series & Spectrum

# READING ASSIGNMENTS

- This Lecture:
  - **Fourier Series in Ch 3, Sect. 3-5**
- Other Reading:
  - Appendix C: More details on Fourier Series



# LECTURE OBJECTIVES

- **Synthesis Summation** of Fourier Series
  - Produces a **PERIODIC** signal: $\mathbf{x(t+T_0) = x(t)}$

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k F_0 t} \quad \text{as } N \to \infty$$

  - Approximation when *N* is finite. How good?
- **SPECTRUM** from Fourier Series
  - $a_k$ is Complex Amplitude for k-th Harmonic



# Harmonic Signal is Periodic

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k F_0 t}$$

*Sums of* ***Harmonic*** *complex exponentials are* ***Periodic*** *signals*

*PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:*

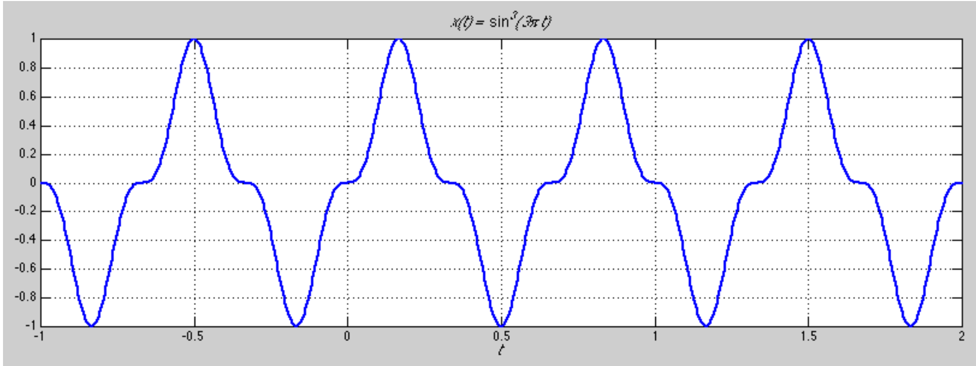
$$2\pi(F_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{F_0}$$

# Fourier Series ANALYSIS

- <u>Some thoughts:</u>
  - Starting from signal, x(t), which frequencies and complex amplitudes are required?
  - ONLY FOR PERIODIC SIGNALS!
  - Two possible analysis methods:
    - **1. Read off coefficients from inverse Euler's**
    - **2. Evaluate Fourier series integral**
  - Can plot the spectrum for the Fourier Series
    - **Equally spaced lines at $kF_0$**



# STRATEGY 1: $x(t)=\sin^3(3\pi t)$



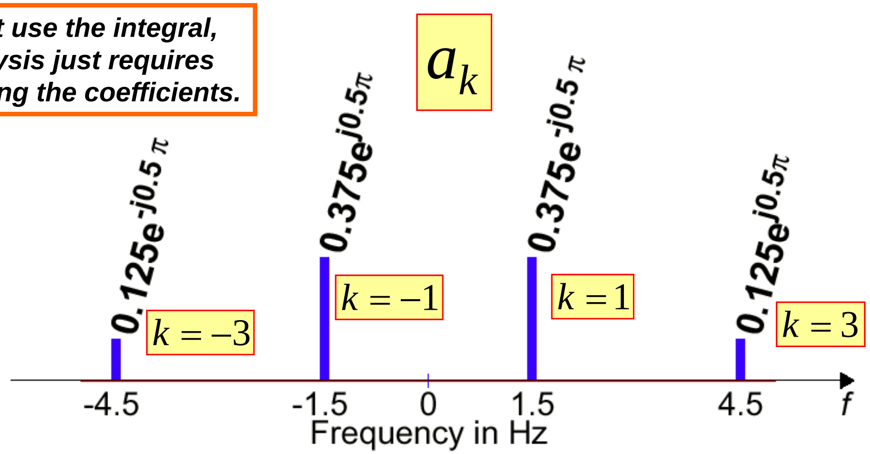

$$x(t)=\left(\frac{j}{8}\right)e^{j9\pi t}+\left(\frac{-3j}{8}\right)e^{j3\pi t}+\left(\frac{3j}{8}\right)e^{-j3\pi t}+\left(\frac{-j}{8}\right)e^{-j9\pi t}$$



# Example $x(t)=\sin^3(3\pi t)$

$$x(t)=\left(\frac{j}{8}\right)e^{j9\pi t}+\left(\frac{-3j}{8}\right)e^{j3\pi t}+\left(\frac{3j}{8}\right)e^{-j3\pi t}+\left(\frac{-j}{8}\right)e^{-j9\pi t}$$

***Don't use the integral, Analysis just requires picking the coefficients.***





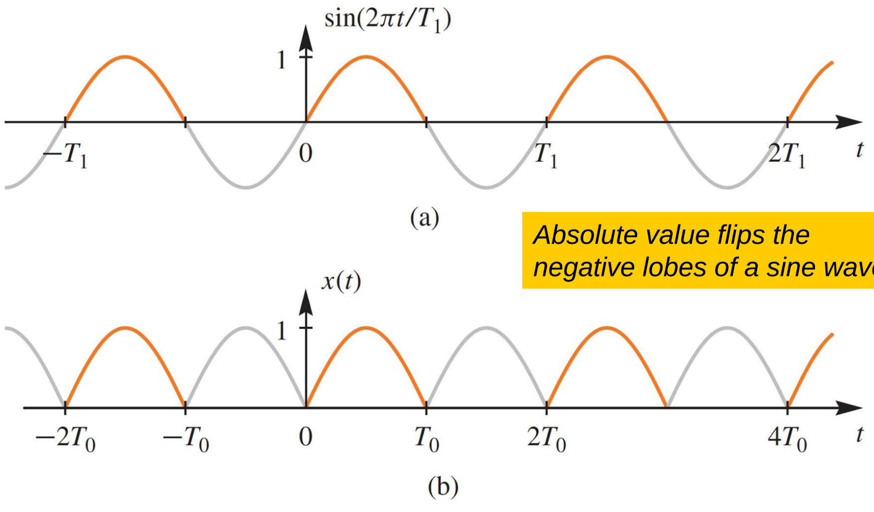

# STRATEGY 2: x(t) → $a_k$

- <u>**ANALYSIS**</u>
  - Get representation from the signal
  - Works for <u>**PERIODIC**</u> Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k=\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}\,dt$$

# Recall FWRS

$x(t) = |\sin(2\pi t / T_1)|$ Period is $T_0 = \frac{1}{2} T_1$




*Absolute value flips the negative lobes of a sine wave*



# FWRS Fourier Integral → {a$_k$}

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt}\,dt$$

***Full-Wave Rectified Sine***

$x(t) = |\sin(2\pi t / T_1)|$

$Period: T_0 = \frac{1}{2} T_1$

$\Rightarrow x(t) = |\sin(\pi t / T_0)|$

$$a_k = \frac{1}{T_0}\int_0^{T_0} \sin(\tfrac{\pi}{T_0} t) e^{-j(2\pi/T_0)kt}\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0} \frac{e^{j(\pi/T_0)t} - e^{-j(\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt}\,dt$$

$$= \left.\frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))}\right|_0^{T_0} - \left.\frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))}\right|_0^{T_0}$$

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$



# FWRS Fourier Coeffs: a$_k$

- $a_k$ is a function of k
  - Complex Amplitude for k-th Harmonic

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$

  - Does not depend on the period, $T_0$
  - DC value is $a_0 = 2/\pi = 0.6336$



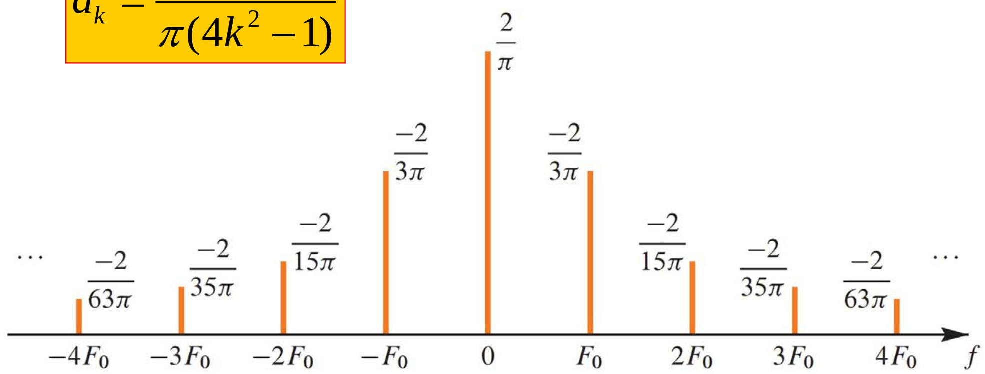


# Spectrum from Fourier Series

Plot $a_k$ for Full-Wave Rectified Sinusoid

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$

$F_0 = 1/T_0$ and $\omega_0 = 2\pi F_0$



# Fourier Series Synthesis

- HOW well do you **APPROXIMATE** x(t) ?

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

- Use FINITE number of coefficients

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k F_0 t}$$

$$a_{-k} = a_k^* \text{ when } x(t) \text{ is real}$$





# Reconstruct From Finite Number of Harmonic Components

Full-Wave Rectified Sinusoid $x(t) = |\sin(\pi t / T_0)|$

$$T_0 = 10\,\text{ms} \Rightarrow F_0 = 100\,\text{Hz}$$

$$a_k = \frac{-2}{\pi(4k^2-1)}$$

$$a_0 = 2/\pi = 0.6336$$

$$x_N(t) = a_0 + \sum_{k=1}^{N}\left\{a_k e^{j2\pi k F_0 t} + a_k^* e^{-j2\pi k F_0 t}\right\}$$

How close is $x_N(t)$ to $x(t) = |\sin(\pi t / T_0)|$?





# Full-Wave Rectified Sine {a_k}

$$a_k = \frac{-2}{\pi(4k^2-1)} \quad \text{is real-valued}$$

$$x_N(t) = \sum_{k=-N}^{N} \frac{-2}{\pi(4k^2-1)} e^{jk\omega_0 t}$$

$$= \frac{-2}{-\pi} + \frac{-2}{\pi(4-1)}e^{j\omega_0 t} + \frac{-2}{\pi(4-1)}e^{-j\omega_0 t} + \frac{-2}{\pi(16-1)}e^{j2\omega_0 t} + \frac{-2}{\pi(16-1)}e^{-j2\omega_0 t} \dots$$

$$= \frac{2}{\pi} - \frac{2}{3\pi}e^{j\omega_0 t} - \frac{2}{3\pi}e^{-j\omega_0 t} - \frac{2}{15\pi}e^{j2\omega_0 t} - \frac{2}{15\pi}e^{-j2\omega_0 t} + \dots$$

$$= \frac{2}{\pi} - \frac{4}{3\pi}\cos(\omega_0 t) - \frac{4}{15\pi}\cos(2\omega_0 t) - \dots - \frac{4}{(4N^2-1)\pi}\cos(N\omega_0 t)$$

- Plots for N=4 and N=9 are shown next
- Excellent Approximation for N=9





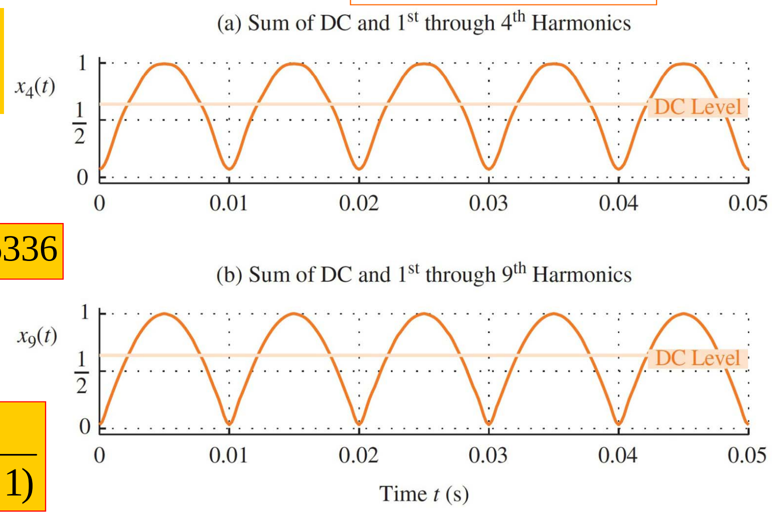
# Reconstruct From Finite Number of Spectrum Components

Full-Wave Rectified Sinusoid $x(t) = |\sin(\pi t / T_0)|$

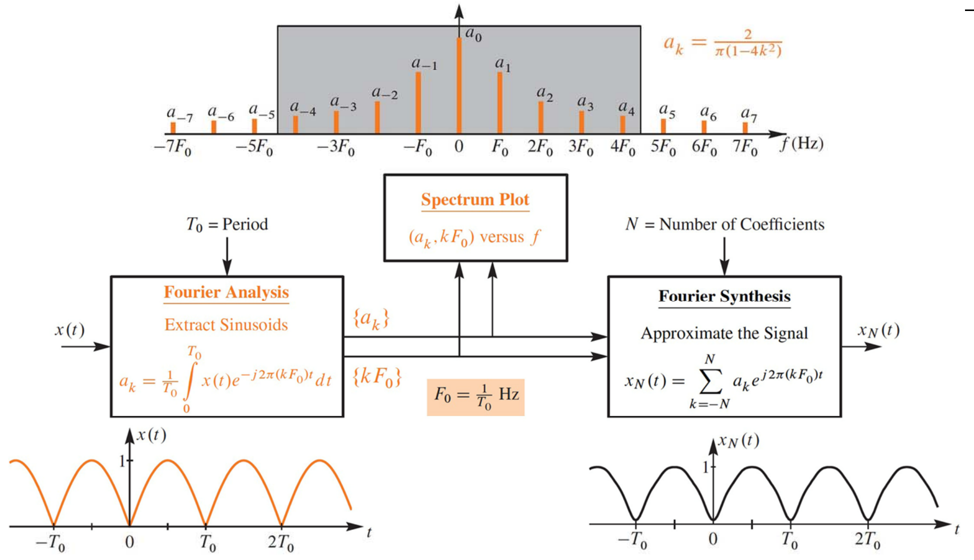
$$T_0 = 10\,\text{ms} \Rightarrow F_0 = 100\,\text{Hz}$$

$$a_0 = 2/\pi = 0.6336$$

$$a_k = \frac{-2}{\pi(4k^2-1)}$$



# Summary of Fourier Series



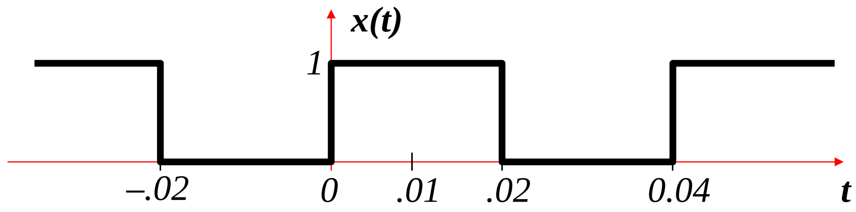





# SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2}T_0 \\ 0 & \frac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.








# FS for a SQUARE WAVE $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k \ne 0)$$

$$x(t) = \begin{cases} 1 & 0 \le t < .02 \\ 0 & .02 \le t < .04 \end{cases}$$

$$a_k = \frac{1}{0.04}\int_0^{.02} 1e^{-j(2\pi/.04)kt}dt = \frac{1}{.04(-j2\pi k/.04)}e^{-j(2\pi/.04)kt}\Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)}(e^{-j(\pi)k} - 1) = \frac{1-(-1)^k}{j2\pi k} \qquad (k \ne 0)$$

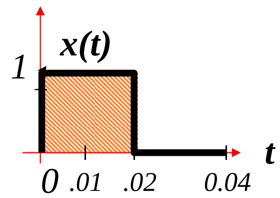




# DC Coefficient: $a_0$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k = 0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt = \frac{1}{T_0}(\text{Area})$$




$$a_0 = \frac{1}{.04}\int_0^{.02} 1dt = \frac{1}{.04}(.02 - 0) = \tfrac{1}{2}$$

# Square Wave Coeffs: $\{a_k\}$

- Complex Amplitude $a_k$ for k-th Harmonic

$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

  - Does not depend on the period, $T_0$
  - DC value is 0.5

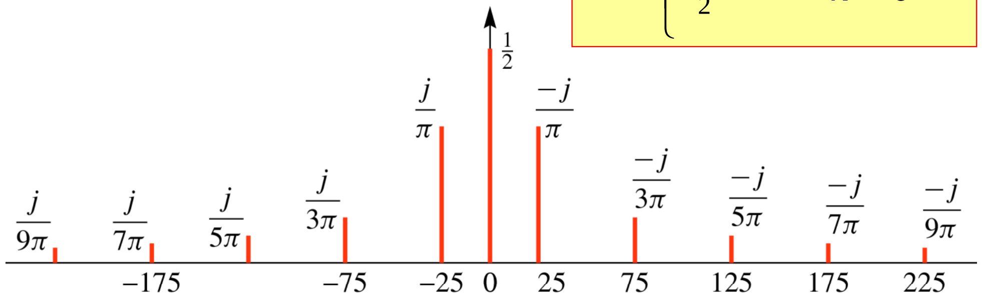




# Spectrum from Fourier Series

$$T_0 = 0.04 \Rightarrow$$
$$\omega_0 = 2\pi/(0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

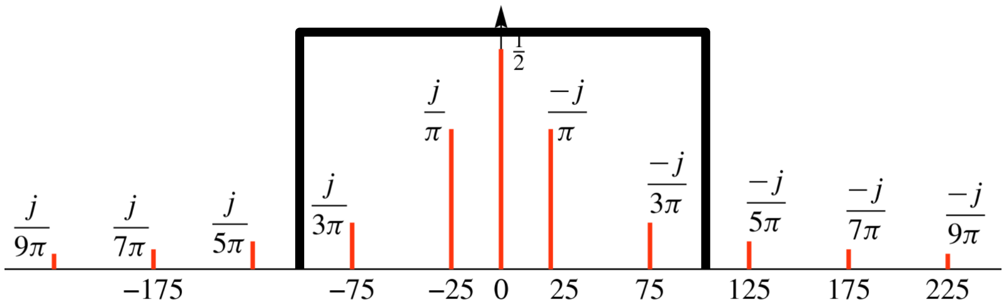


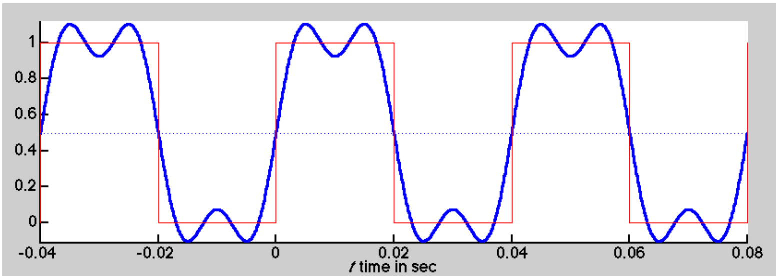





# Synthesis: 1st & 3rd Harmonics

$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi(25)t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\cos(2\pi(75)t - \tfrac{\pi}{2})$$



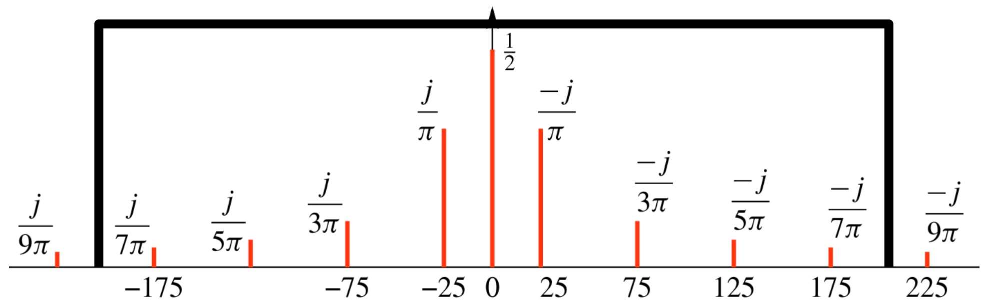



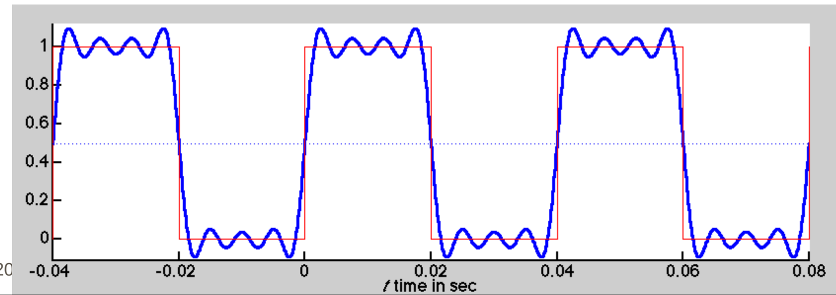




# Synthesis: up to 7th Harmonic

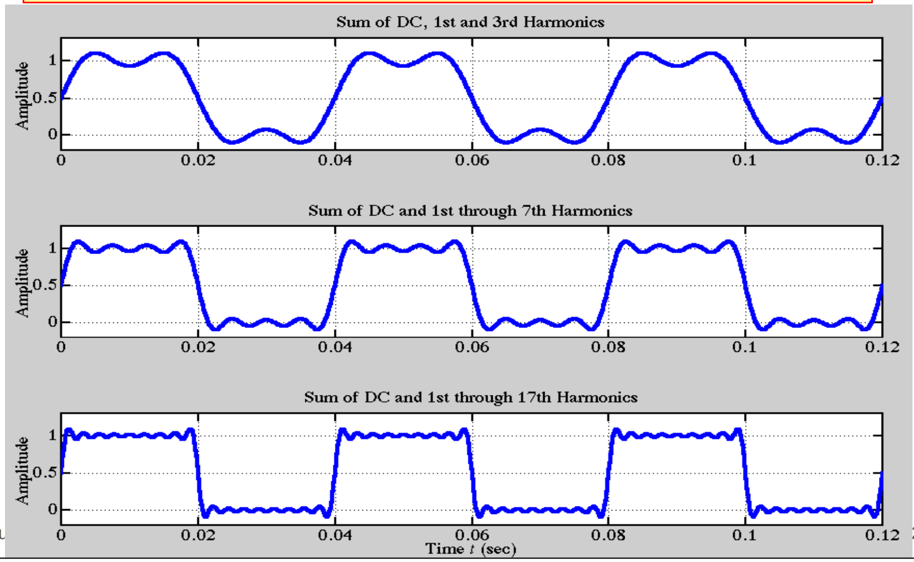
$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(50\pi t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\sin(150\pi t) + \frac{2}{5\pi}\sin(250\pi t) + \frac{2}{7\pi}\sin(350\pi t)$$



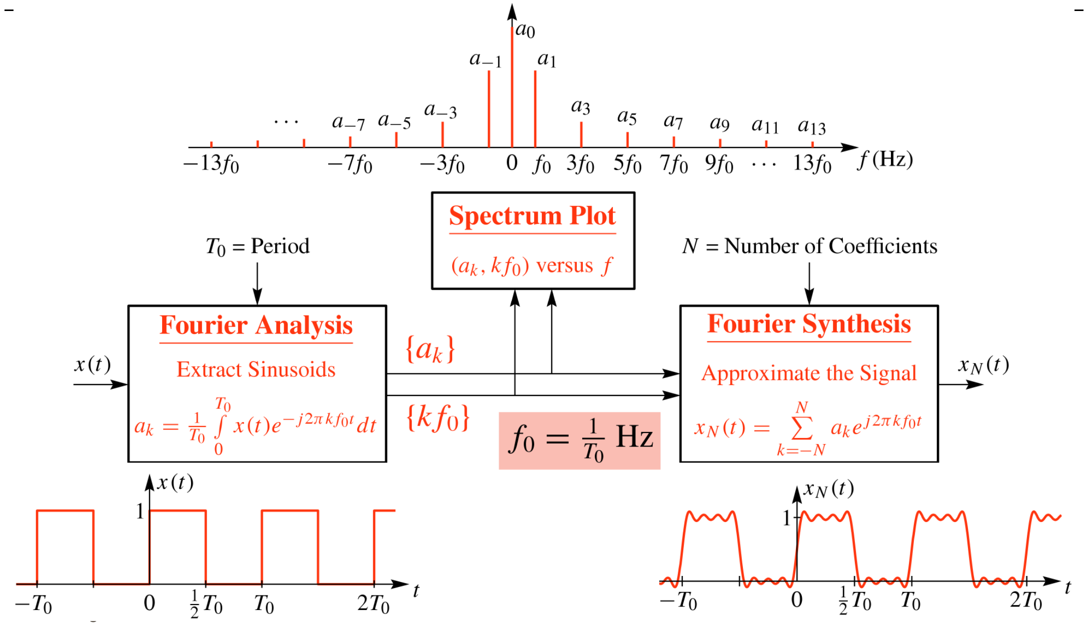
# Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \ldots$$



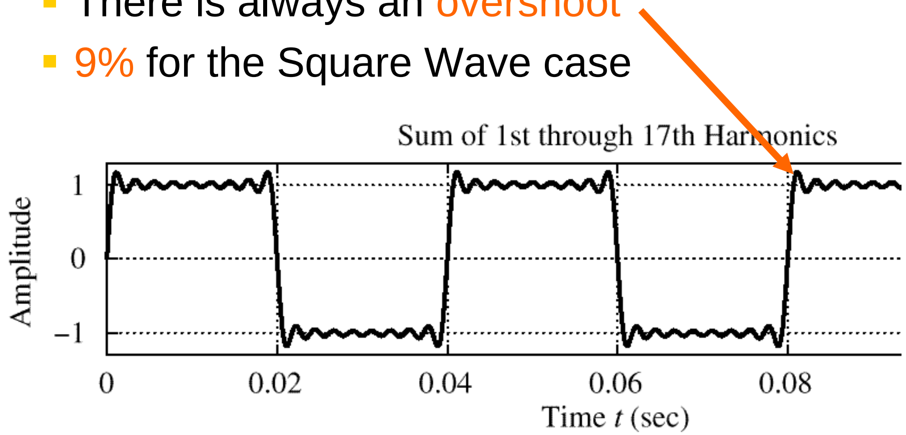



# Fourier Series Summary




# Gibbs' Phenomenon

- Convergence at DISCONTINUITY of x(t)
  - There is always an overshoot
  - 9% for the Square Wave case








# Fourier Series Demos

- MATLAB GUI: fseriesdemo
  - Shows the convergence with more terms
  - One of the demos in:
  - http://dspfirst.gatech.edu/matlab/

# fseriesdemo GUI