DSP First, 2/e

Lecture 9

Sampling & Aliasing

READING ASSIGNMENTS

- This Lecture:
  - Chap 4, Sections 4-1 and 4-2
- Other Reading:
  - Recitation: Strobe Demo (Sect 4-5)
  - Next Lecture: Chap. 4, Sects. 4-3 and 4-4

LECTURE OBJECTIVES

- SAMPLING can cause ALIASING
  - **Sampling Theorem**
  - Sampling Rate > 2(Highest Frequency)
- Spectrum for digital signals, x[n]
  - Normalized Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi\ell$$

ALIASING

SYSTEMS Process Signals

- PROCESSING GOALS:
  - Change x(t) into y(t)
    - For example, more BASS, pitch shifting
  - Improve x(t), e.g., image deblurring
  - Extract Information from x(t)

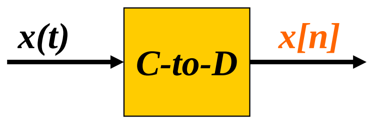
System IMPLEMENTATION

- *ANALOG/ELECTRONIC:*
  - *Circuits: resistors, capacitors, op-amps*

- DIGITAL/MICROPROCESSOR
  - Convert x(t) to numbers stored in memory

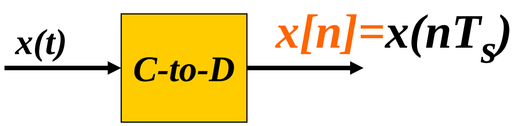
SAMPLING x(t)

- SAMPLING PROCESS
  - Convert x(t) to numbers x[n]
  - "n" is an integer index; x[n] is a sequence of values
  - Think of "n" as the storage address in memory
- UNIFORM SAMPLING at $t = nT_s$
  - IDEAL: $x[n] = x(nT_s)$



SAMPLING RATE, f_s

- SAMPLING RATE (f_s)
  - $f_s = 1/T_s$
    - NUMBER of SAMPLES PER SECOND
  - T_s = 125 microsec → f_s = 8000 samples/sec
    - UNITS of f_s ARE HERTZ: 8000 Hz
- UNIFORM SAMPLING at $t = nT_s = n/f_s$
  - IDEAL: $x[n] = x(nT_s) = x(n/f_s)$



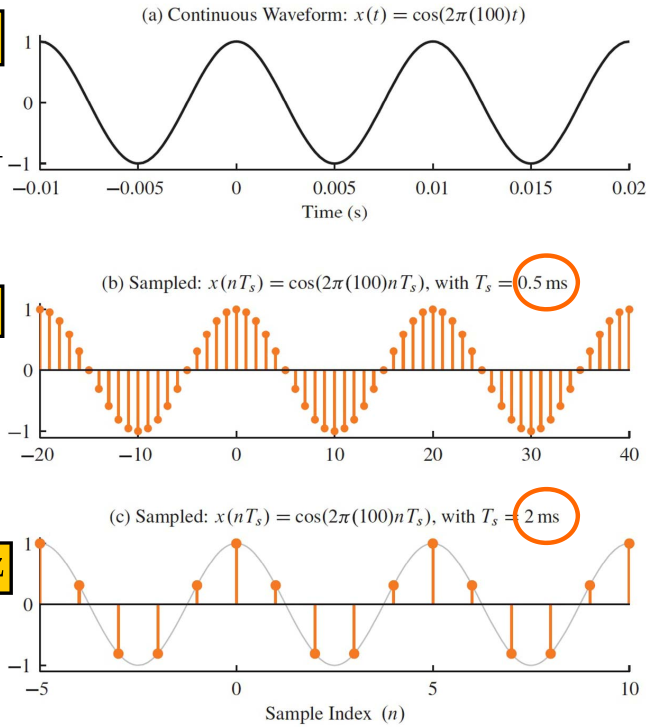
STORING DIGITAL SOUND

- ***x[n]*** is a SAMPLED SISIGNAL
  - A list of numbers stored in memory
- EXAMPLE: audio CD
- CD rate is 44,100 samples per second
  - 16-bit samples
  - Stereo uses 2 channels
- Number of bytes for 1 minute is
  - 2 X (16/8) X 60 X 44100 = 10.584 Mbytes

$f = 100\text{Hz}$

$f_s = 2\,\text{kHz}$

$f_s = 500\text{Hz}$



Which one provides the most accurate representation of x(t)?

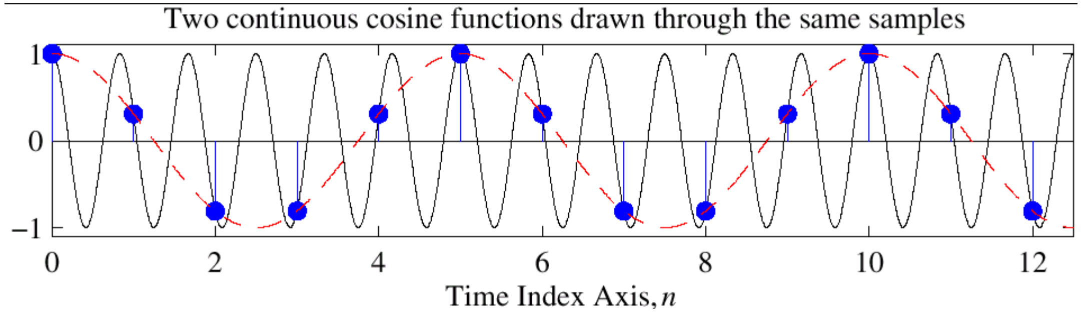
SAMPLING THEOREM

- HOW OFTEN DO WE NEED TO SAMPLE?
  - DEPENDS on FREQUENCY of SINUSOID
  - ANSWERED by SHANNON/NYQUIST Theorem
  - ALSO DEPENDS on "**RECONSTRUCTION**"

> *Shannon Sampling Theorem*
> A continuous-time signal $x(t)$ with frequencies no higher than $f_{\max}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\max}$.

Reconstruction? Which One?

Given the samples, draw a sinusoid through the values



$$x[n] = \cos(0.4\pi n)$$

When n is an integer

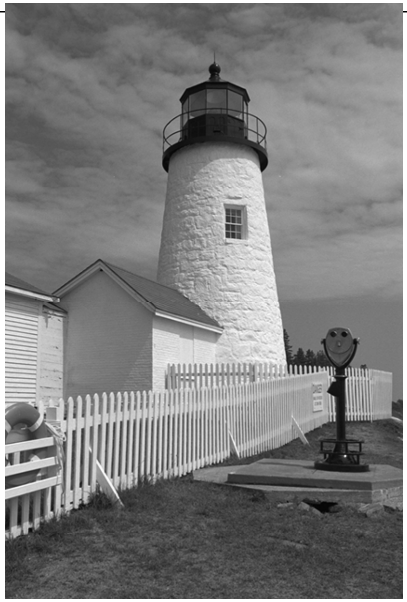
$$\cos(0.4\pi n) = \cos(2.4\pi n)$$

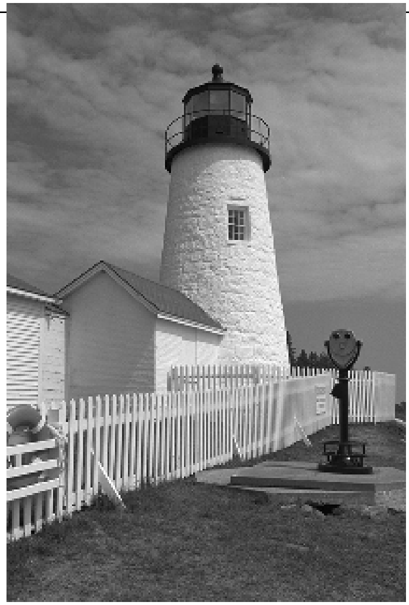
Occam's razor -> pick lowest frequency sinusoid

Be careful...

https://www.youtube.com/watch?v=qgvuQGY946g

Spatial Aliasing





DISCRETE-TIME SINUSOID

- Change x(t) into x[n] **DERIVATION**

$$x(t) = A\cos(\omega t + \varphi)$$

$$x[n] = x(nT_s) = A\cos(\omega n T_s + \varphi)$$

$$x[n] = A\cos((\omega T_s) n + \varphi)$$

$$x[n] = A\cos(\hat{\omega} n + \varphi)$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

DEFINE DIGITAL FREQUENCY

DIGITAL FREQUENCY $\hat{\omega}$

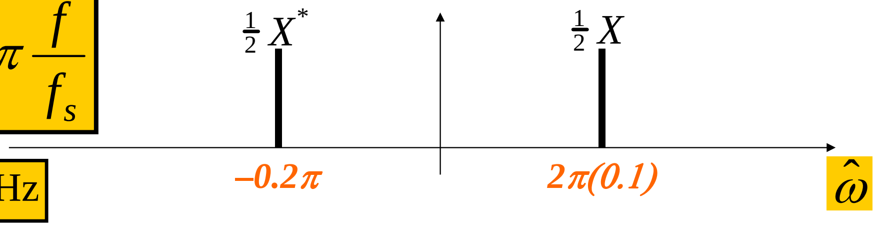
$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}$$

- $\hat{\omega}$ VARIES from **0** to **2π**, as f varies from 0 to the sampling frequency
- UNITS are radians, **not** rad/sec
  - DIGITAL FREQUENCY is NORMALIZED

SPECTRUM (DIGITAL)

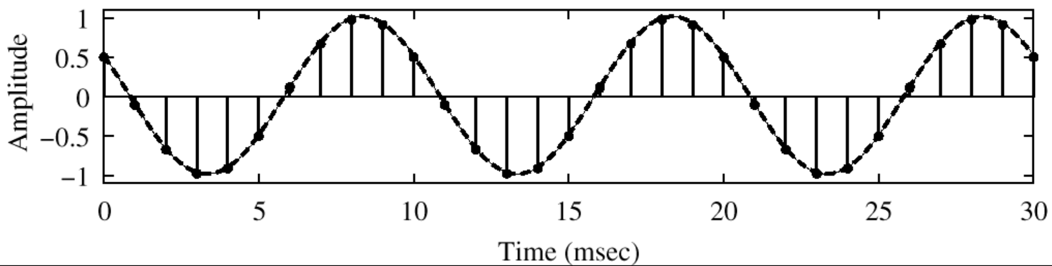
$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$f_s = 1\,\text{kHz}$



$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$

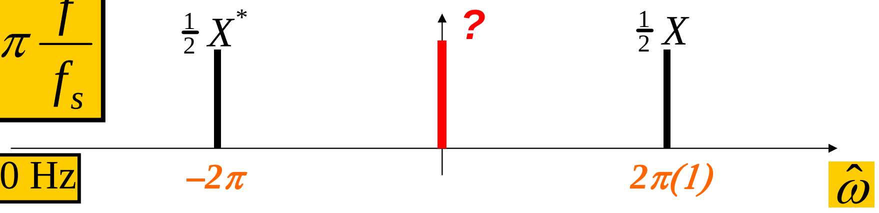
100-Hz Cosine Wave: Sampled with T_s = 1 msec (1000 Hz)



SPECTRUM (DIGITAL) ???

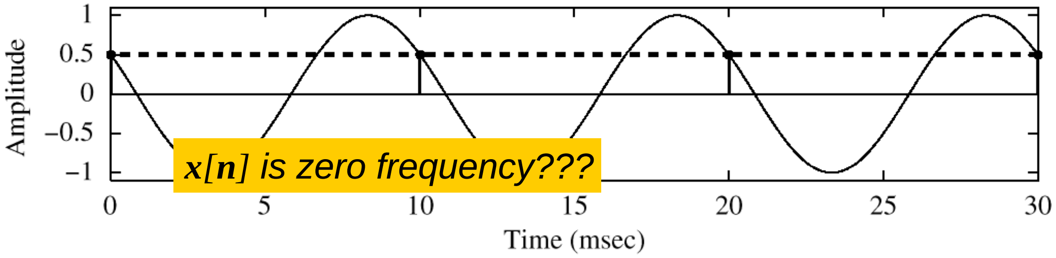
$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$f_s = 100$ Hz



$$x[n] = A\cos(2\pi(100)(n/100) + \varphi)$$

100-Hz Cosine Wave: Sampled with T_s = 10 msec (100 Hz)



The REST of the STORY

- Spectrum of x[n] has more than one line for each complex exponential
  - Called **ALIASING**
  - **MANY SPECTRAL LINES**
- SPECTRUM is PERIODIC with period = **2π**
  - Because

$$A\cos(\hat{\omega}n + \varphi) = A\cos((\hat{\omega} + 2\pi\ell)n + \varphi)$$

ALIASING DERIVATION

- Other Frequencies give the same $\hat{\omega}$

$x_1(t) = \cos(400\pi t)$ sampled at $f_s = 1000$ Hz

$$x_1[n] = \cos(400\pi \tfrac{n}{1000}) = \cos(0.4\pi n)$$

$x_2(t) = \cos(2400\pi t)$ sampled at $f_s = 1000$ Hz

$$x_2[n] = \cos(2400\pi \tfrac{n}{1000}) = \cos(2.4\pi n)$$

$$x_2[n] = \cos(2.4\pi n) = \cos(0.4\pi n + 2\pi n) = \cos(0.4\pi n)$$

$$\Rightarrow x_2[n] = x_1[n] \qquad 2400\pi - 400\pi = 2\pi(1000)$$

ALIASING DERIVATION

- Other Frequencies give the same $\hat{\omega}$

If $x(t) = A\cos(2\pi(f + \ell f_s)t + \varphi)$ $\qquad t \leftarrow \frac{n}{f_s}$

and we want : $x[n] = A\cos(\hat{\omega}n + \varphi)$

$$\text{then : } \hat{\omega} = \frac{2\pi(f + \ell f_s)}{f_s} = \frac{2\pi f}{f_s} + \frac{2\pi \ell f_s}{f_s}$$

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} \quad + 2\pi\ell$$

ALIASING CONCLUSIONS

- Adding an **<u>INTEGER multiple</u>** of f_s or $-f_s$ to the frequency of a continuous sinusoid $x_c(t)$ gives **<u>exactly the same values</u>** for the sampled signal $x[n] = x_c(n/f_s)$
- **GIVEN x[n], we CAN'T KNOW whether it came from a sinusoid at f_o or $(f_o + f_s)$ or $(f_o + 2f_s)$ …**
- **This is called ALIASING**

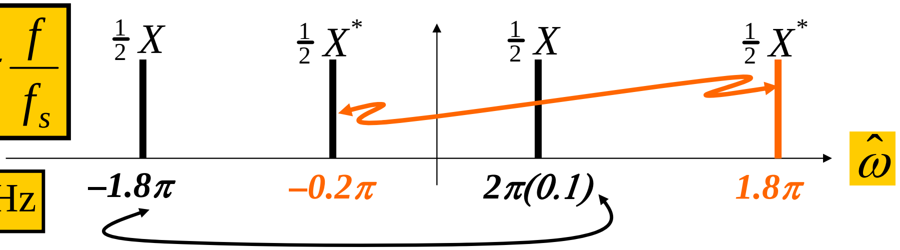
SPECTRUM for x[n]

- PLOT versus NORMALIZED FREQUENCY
- INCLUDE <u>ALL</u> SPECTRUM LINES
  - ALIASES
    - ADD MULTIPLES of 2π
    - SUBTRACT MULTIPLES of 2π
  - FOLDED ALIASES
    - (to be discussed later)
    - ALIASES of NEGATIVE FREQS

SPECTRUM (MORE LINES)

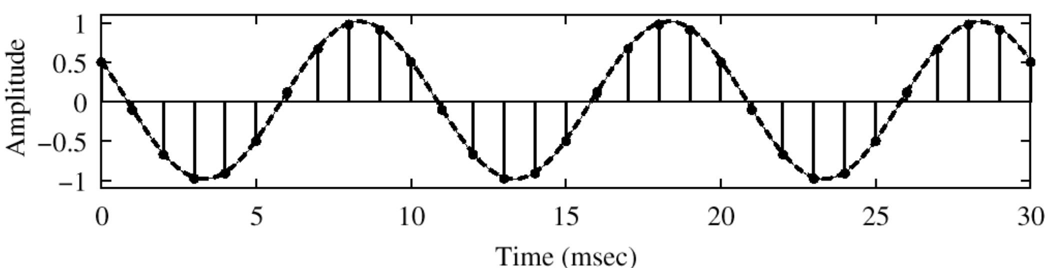
$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 1\,\text{kHz}$$



$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$

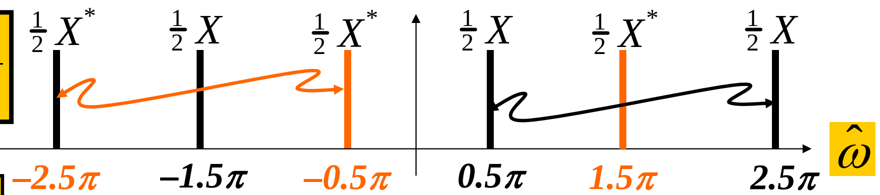
100-Hz Cosine Wave: Sampled with T_s = 1 msec (1000 Hz)



SPECTRUM (ALIASING CASE)

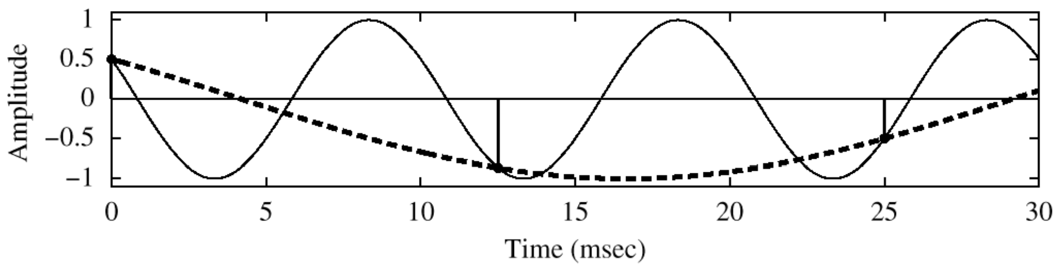
$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 80\,\text{kHz}$$

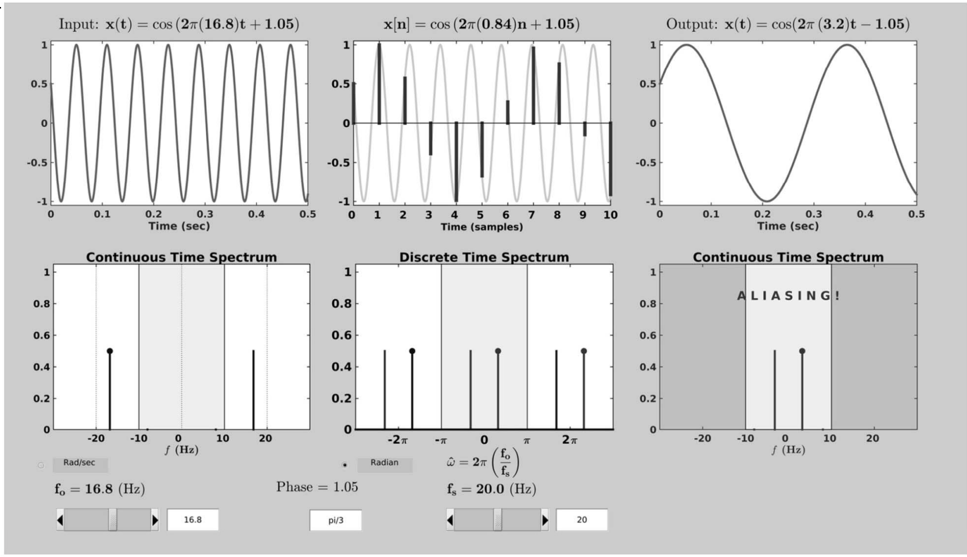


$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$

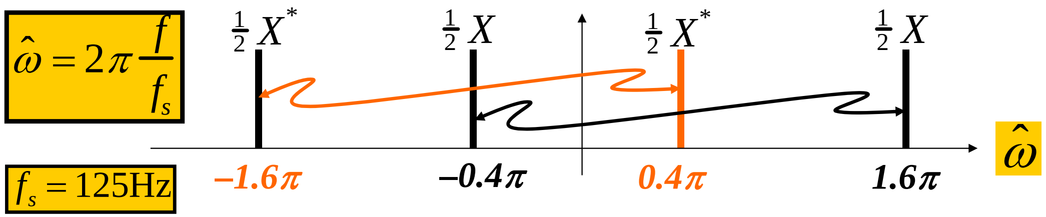
100-Hz Cosine Wave: Sampled with T_s = 12.5 msec (80 Hz)



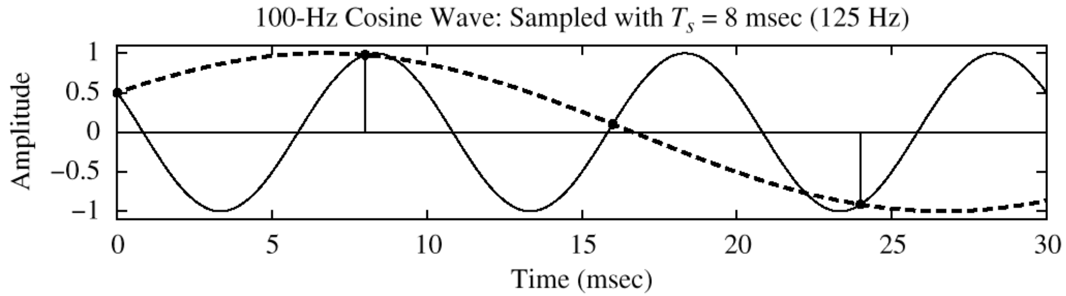
SAMPLING GUI (con2dis)



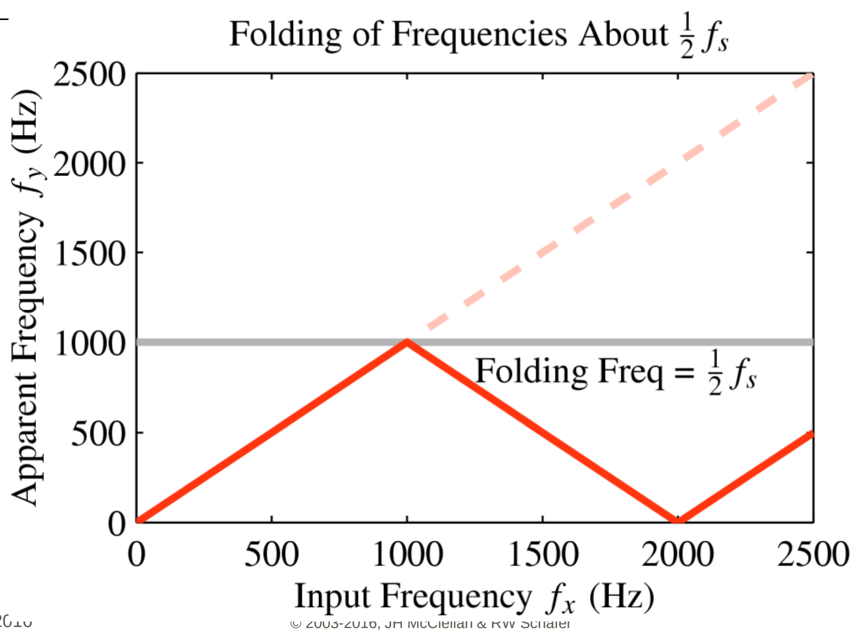
SPECTRUM (FOLDING CASE)



$$x[n] = A\cos(2\pi(100)(n/125)+\varphi)$$



FOLDING DIAGRAM



STROBE DEMO (Synthetic)