DSP First, 2/e

Lecture 10

Digital to Analog (D-to-A) Conversion

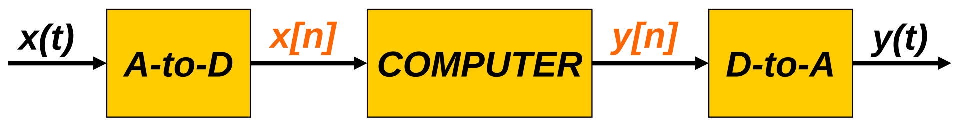
READING ASSIGNMENTS

- This Lecture:
  - Chapter 4: Sections 4-3, 4-4
- Other Reading:
  - Recitation: Section 4-5 (Strobe Demo)
    - Next Lecture: Chapter 5 (beginning)

LECTURE OBJECTIVES

- FOLDING: a type of ALIASING
- DIGITAL-to-ANALOG CONVERSION is
  - Reconstruction from samples
    - SAMPLING THEOREM applies
  - Smooth **Interpolation**
- Mathematical Model of D-to-A
  - SUM of SHIFTED PULSES
    - Linear Interpolation example

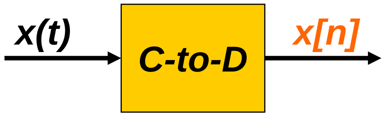
SIGNAL TYPES



- A-to-D
  - Convert x(t) to numbers stored in memory
- D-to-A
  - Convert y[n] back to a "continuous-time" signal, y(t)
  - y[n] is called a "discrete-time" signal

SAMPLING x(t)

- UNIFORM SAMPLING at $t = nT_s$
  - IDEAL: $x[n] = x(nT_s)$



Shannon Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than $f_{\max}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\max}$.

Terminology: NYQUIST RATE

- "**Nyquist Rate**" Sampling
  - f_s > **TWICE** the HIGHEST Frequency in x(t)
  - "Sampling above the Nyquist rate"
- **BANDLIMITED SIGNALS**
  - DEF: HIGHEST FREQUENCY COMPONENT in SPECTRUM of x(t) is finite
  - NON-BANDLIMITED EXAMPLE
    - TRIANGLE WAVE is NOT BANDLIMITED

SPECTRUM for x[n]

- INCLUDE ALL SPECTRUM LINES
  - ALIASES
    - ADD INTEGER MULTIPLES of 2π and -2π
  - FOLDED ALIASES
    - ALIASES of NEGATIVE FREQS
- PLOT versus NORMALIZED FREQUENCY
  - i.e., DIVIDE f_o by f_s

$$\hat{\omega} = 2\pi \frac{(\pm f_0)}{f_s} + 2\pi\ell$$

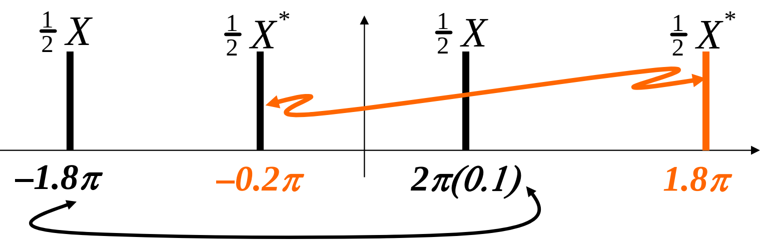
EXAMPLE: SPECTRUM

- $x[n] = A\cos(0.2\pi n+\phi)$
- FREQS @ 0.2π and -0.2π
- ALIASES:
  - $\{2.2\pi, 4.2\pi, 6.2\pi, \ldots\}$ & $\{-1.8\pi, -3.8\pi, \ldots\}$
  - EX: $x[n] = A\cos(4.2\pi n+\phi)$
- ALIASES of NEGATIVE FREQ:
  - $\{1.8\pi, 3.8\pi, 5.8\pi, \ldots\}$ & $\{-2.2\pi, -4.2\pi \ldots\}$

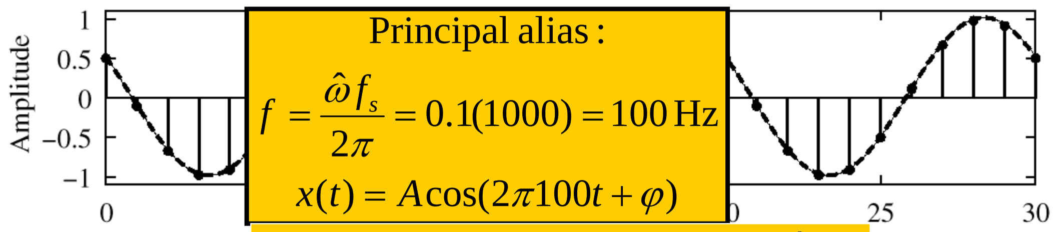
SPECTRUM (MORE LINES)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$f_s = 1\,\text{kHz}$



$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$



Principal alias:

$$f = \frac{\hat{\omega} f_s}{2\pi} = 0.1(1000) = 100\,\text{Hz}$$

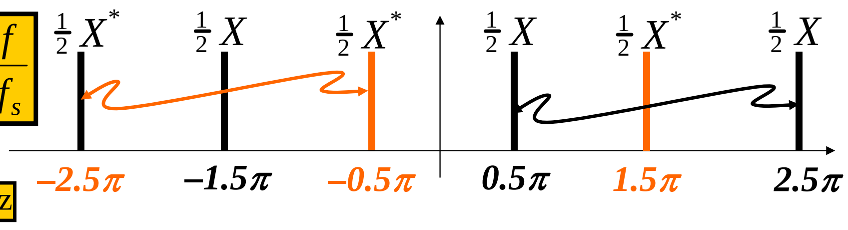
$$x(t) = A\cos(2\pi 100 t + \varphi)$$

Principal alias is always between $-\pi \le \hat{\omega} \le \pi$

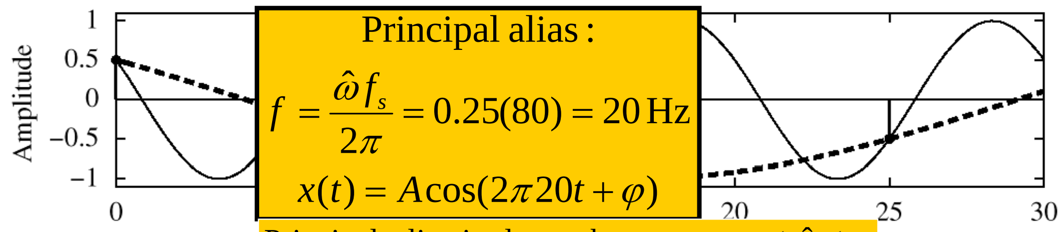
SPECTRUM (ALIASING CASE)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$f_s = 80\,\text{Hz}$



$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$



Principal alias:

$$f = \frac{\hat{\omega} f_s}{2\pi} = 0.25(80) = 20\,\text{Hz}$$

$$x(t) = A\cos(2\pi 20 t + \varphi)$$

Principal alias is always between $-\pi \le \hat{\omega} \le \pi$

FOLDING (a type of ALIASING)

- EXAMPLE: 3 different x(t); same x[n]

$$\hat{\omega} = 2\pi \frac{100}{1000} = 2\pi(0.1)$$

$$f_s = 1000$$

$$\cos(2\pi(100)t) \rightarrow \cos[2\pi(0.1)n]$$

$$\cos(2\pi(1100)t) \rightarrow \cos[2\pi(1.1)n] = \cos[2\pi(0.1)n]$$

$$\cos(2\pi(900)t) \rightarrow \cos[2\pi(0.9)n]$$

$$= \cos[2\pi(0.9)n - 2\pi n] = \cos[2\pi(-0.1)n] = \cos[2\pi(0.1)n]$$

- *900 Hz "folds" to 100 Hz when f_s=1kHz*

DIGITAL FREQ $\hat{\omega}$ AGAIN

Normalized Radian Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi\ell$$

ALIASING

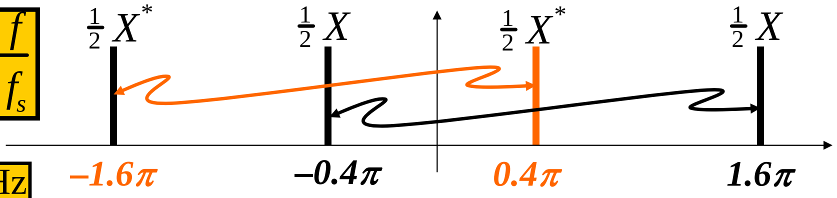
$$\hat{\omega} = \omega T_s = -\frac{2\pi f}{f_s} + 2\pi\ell$$

FOLDED ALIAS

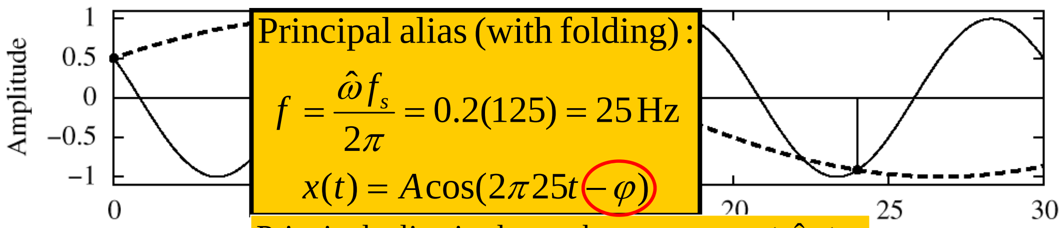
SPECTRUM (FOLDING CASE)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 125\text{Hz}$$



$$x[n] = A\cos(2\pi(100)(n/125) + \varphi)$$



Principal alias (with folding):

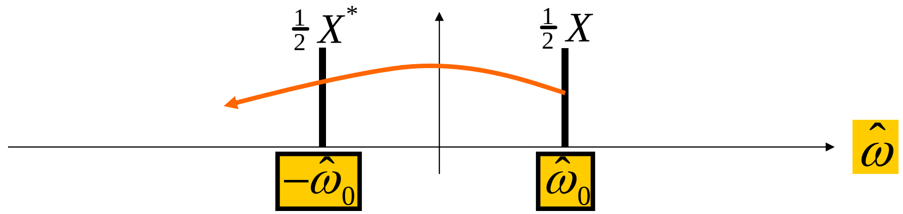
$$f = \frac{\hat{\omega} f_s}{2\pi} = 0.2(125) = 25\,\text{Hz}$$

$$x(t) = A\cos(2\pi 25 t - \varphi)$$

Principal alias is always between $-\pi \le \hat{\omega} \le \pi$

SPECTRUM Explanation of SAMPLING THEOREM

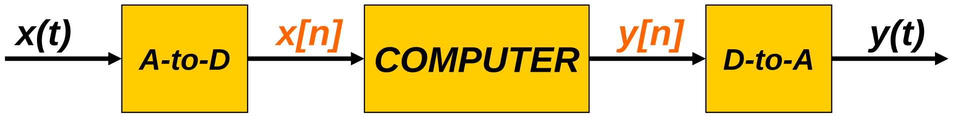
- How do we prevent aliasing?
- Guarantee original signal is principal alias:



$$\hat{\omega}_0 - 2\pi < -\hat{\omega}_0 \Rightarrow \hat{\omega}_0 < \pi$$

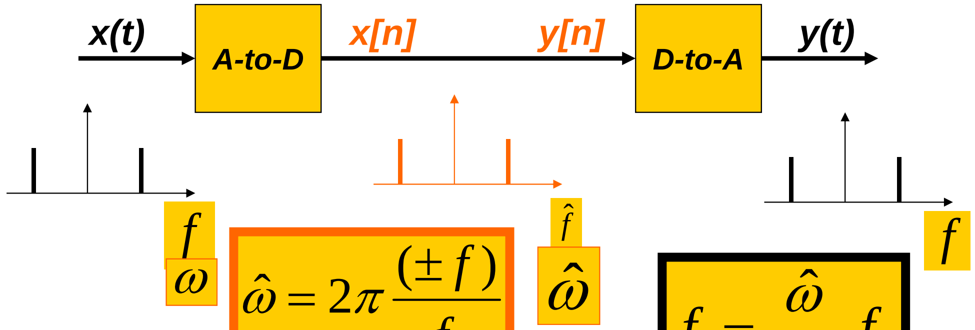
$$\hat{\omega}_0 = \frac{2\pi f_0}{f_s} < \pi \Rightarrow f_0 < \frac{f_s}{2}$$

D-to-A Reconstruction



- Create continuous y(t) from y[n]
  - **IDEAL D-to-A:**
    - If you have formula for y[n]
  - Invert sampling ($t=nT_s$) by $\mathbf{n=f_s t}$
  - y[n] = Acos(0.2πn+ϕ)   with f_s = 8000 Hz
  - y(t) = Acos(0.2π(8000t)+ϕ) = Acos(2π(**800**)t+ϕ)

FREQUENCY DOMAINS



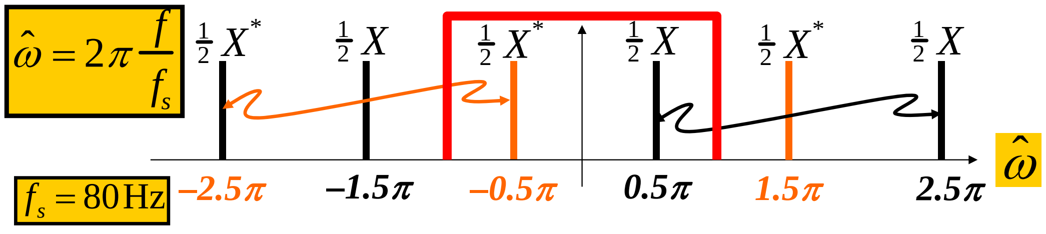
$$\hat{\omega} = 2\pi \frac{(\pm f)}{f_s} + 2\pi\ell$$

$$f = \frac{\hat{\omega}}{2\pi} f_s$$

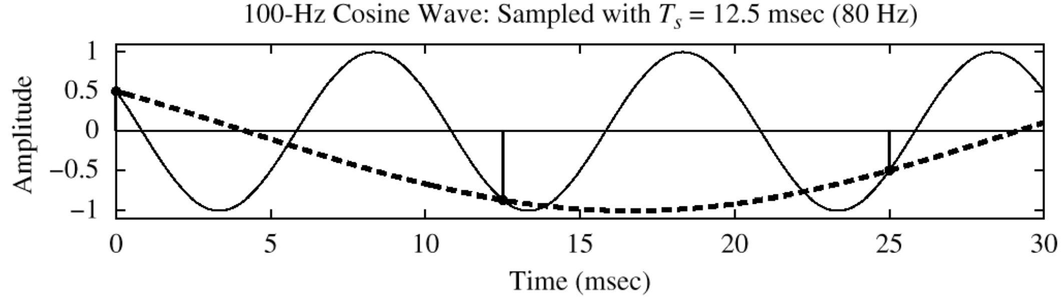
D-to-A is AMBIGUOUS !

- ALIASING
  - Given y[n], which y(t) do we pick ? ? ?
  - INFINITE NUMBER of y(t)
    - PASSING THRU THE SAMPLES, y[n]
  - D-to-A RECONSTRUCTION MUST CHOOSE ONE OUTPUT
- RECONSTRUCT THE SMOOTHEST ONE
  - THE LOWEST FREQ, if y[n] = sinusoid

SPECTRUM (ALIASING CASE)



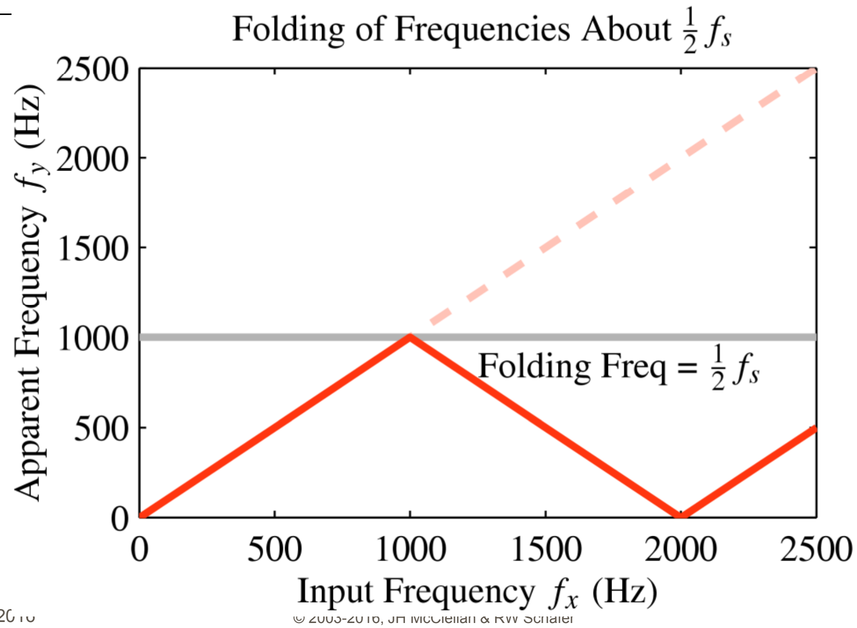
$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$



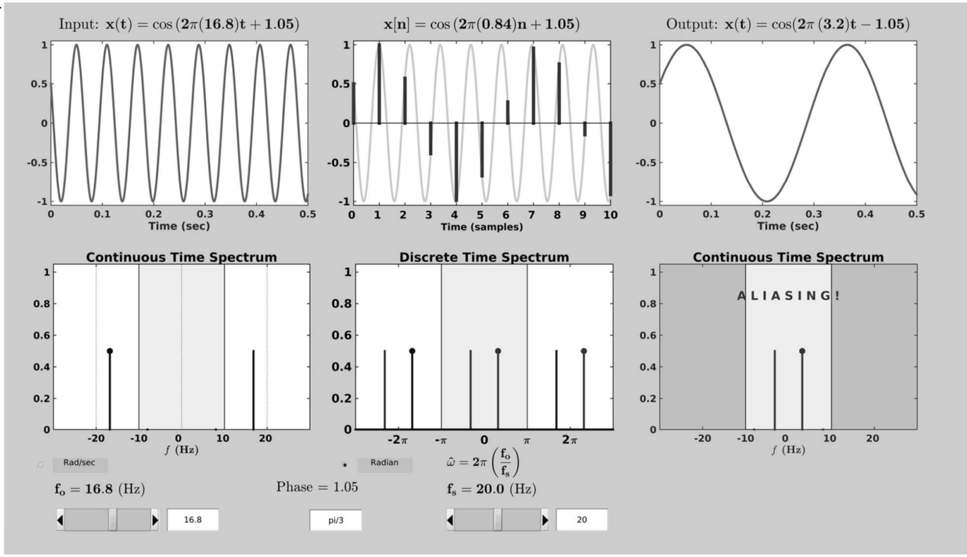
DEMOS from CHAPTER 4

- CD-ROM DEMOS
- SAMPLING DEMO (**con2dis GUI**)
  - Different Sampling Rates
    - Aliasing of a Sinusoid
- STROBE DEMO
  - Synthetic vs. Real
  - Television SAMPLES at 30 fps
- Sampling & Reconstruction

FOLDING DIAGRAM $f_s = 2000\,\text{Hz}$

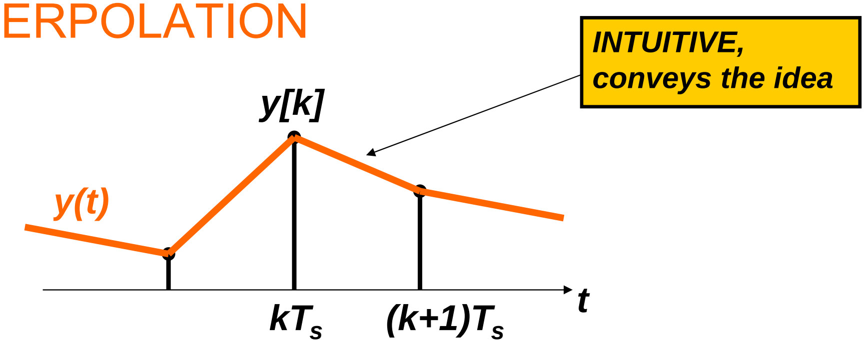


SAMPLING GUI (con2dis)



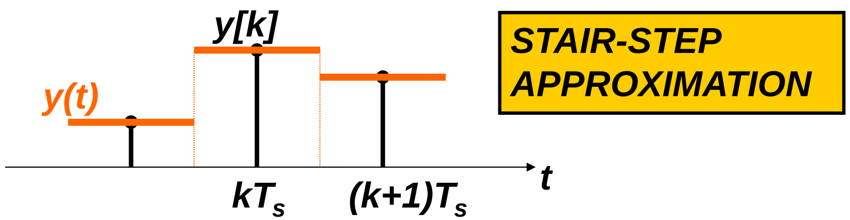
Reconstruction (D-to-A)

- CONVERT STREAM of NUMBERS to x(t)
- "CONNECT THE DOTS"
- INTERPOLATION

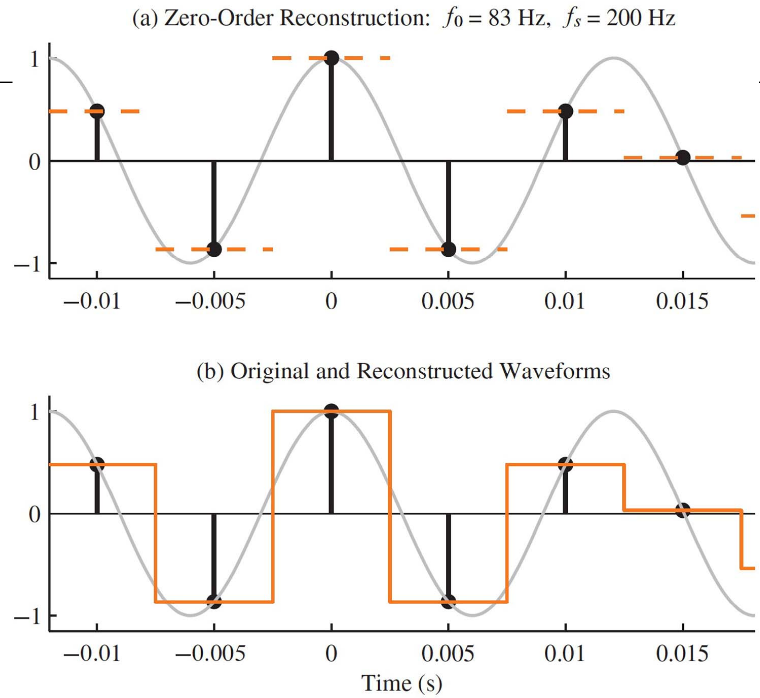


SAMPLE & HOLD DEVICE

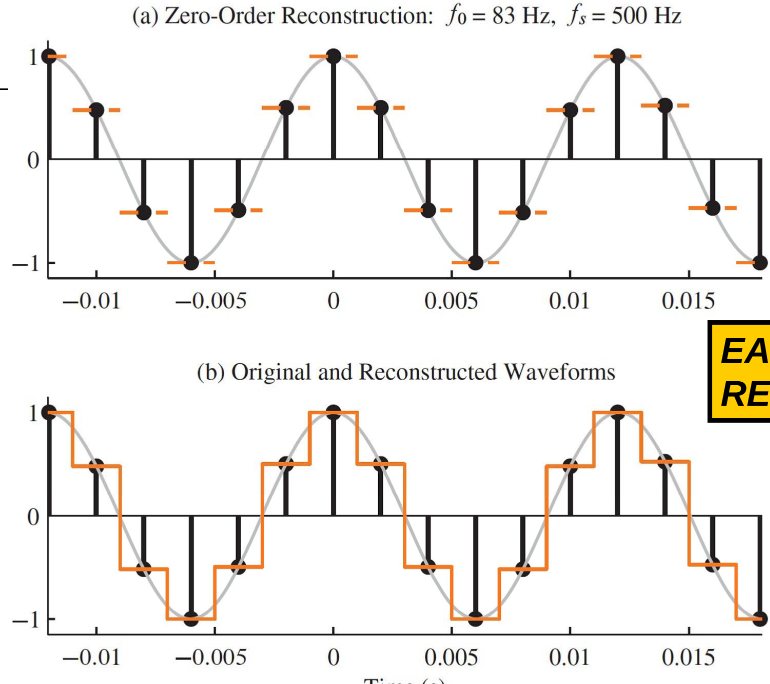
- CONVERT y[n] to y(t)
  - y[k] should be the value of y(t) at $t = kT_s$
  - Make y(t) equal to y[k] for
    - $kT_s - 0.5T_s < t < kT_s + 0.5T_s$



SQUARE PULSE CASE



OVER-SAMPLING CASE



EASIER TO RECONSTRUCT

MATH MODEL for D-to-A

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$

SQUARE PULSE:

$$p(t) = \begin{cases} 1 & -\frac{1}{2}T_s < t \le \frac{1}{2}T_s \\ 0 & \text{otherwise} \end{cases}$$

EXPAND the SUMMATION

$$\sum_{n=-\infty}^{\infty} y[n]p(t - nT_s) =$$

$$\ldots + y[0]p(t) + y[1]p(t - T_s) + y[2]p(t - 2T_s) + \ldots$$

- SUM of SHIFTED PULSES $p(t-nT_s)$
  - "WEIGHTED" by y[n]
  - CENTERED at $t=nT_s$
  - SPACED by T_s
    - RESTORES "REAL TIME"

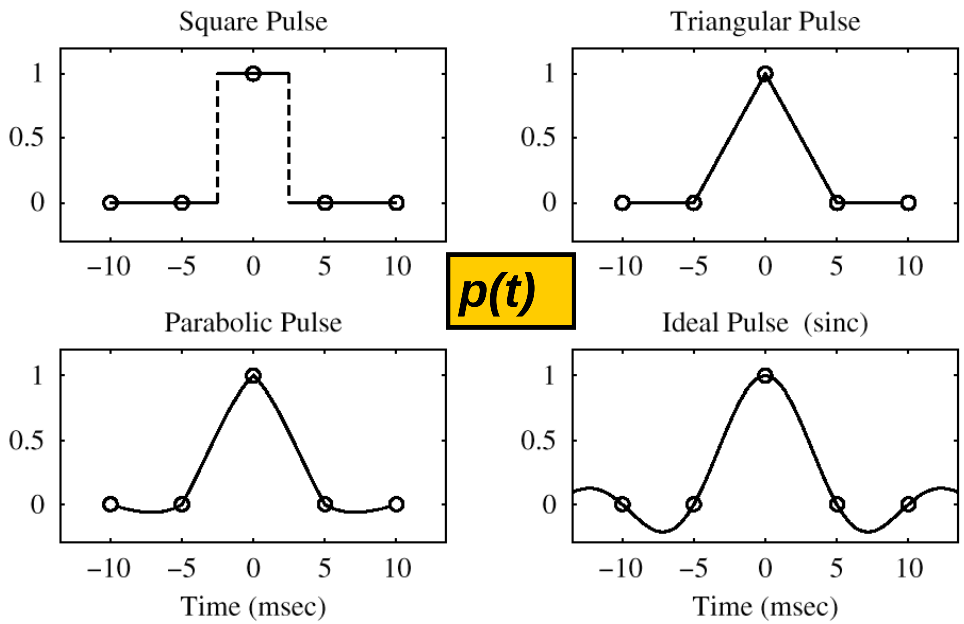
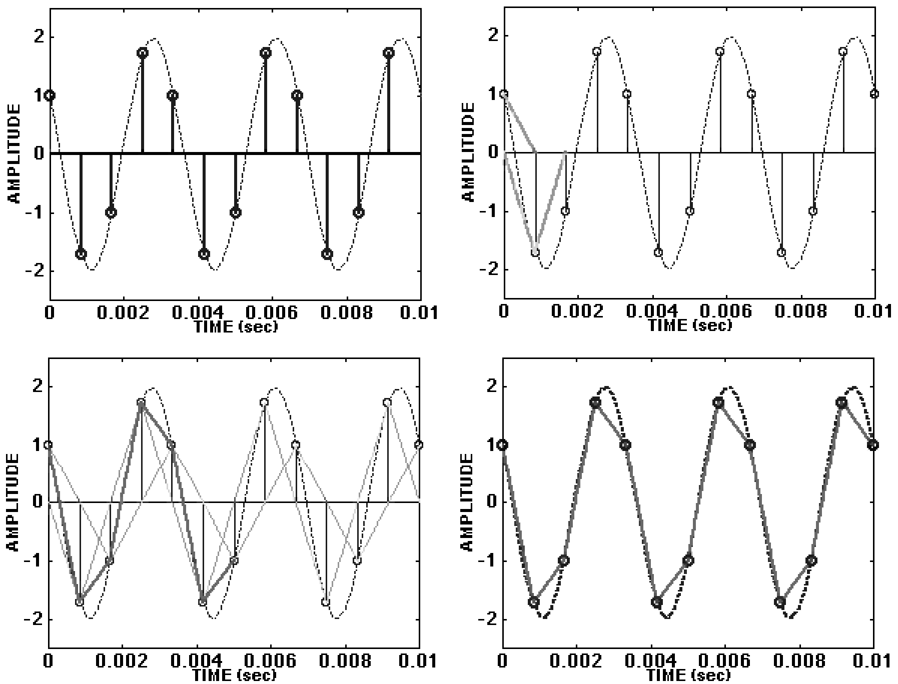


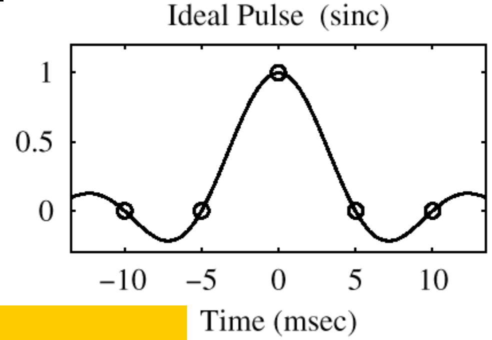
Figure 4.17 Four different pulses for D-to-C conversion. The sampling period is $T_s = 0.005$, i.e., $f_s = 200$ Hz. Note that the duration of each pulse is approximately one or two times T_s.

TRIANGULAR PULSE (2X)



OPTIMAL PULSE ?

CALLED "BANDLIMITED INTERPOLATION"



$$p(t) = \frac{\sin \frac{\pi t}{T_s}}{\frac{\pi t}{T_s}} \quad \text{for } -\infty < t < \infty$$

$$p(t) = 0 \quad \text{for } t = \pm T_s, \pm 2T_s, \ldots$$

Reconstruct with Ideal (2x)