DSP First, 2/e

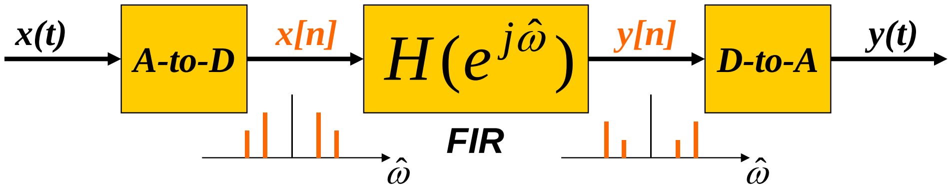
Lecture 14
Digital Filtering
of Analog Signals

READING ASSIGNMENTS

- This Lecture:
  - Chapter 6, Sections 6-6, 6-7 & 6-8
  - Next Lecture: Chapter 7 (DTFT)

LECTURE OBJECTIVES

- Two Domains: Time & Frequency
- Track the spectrum of x[n] thru an FIR Filter:
  Sinusoid-IN gives Sinusoid-OUT
- **UNIFICATION**: How does the Frequency Response affect x(t) to produce y(t) ?



TIME & FREQUENCY

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k] x[n-k]$$

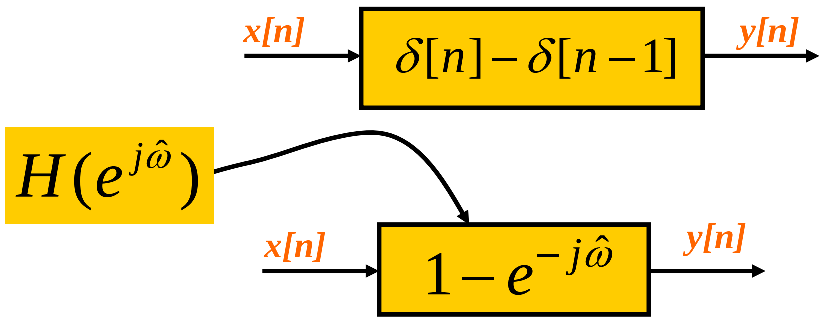
FIR DIFFERENCE EQUATION is the TIME-DOMAIN

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} h[k] e^{-j\hat{\omega}k}$$

$$H(e^{j\hat{\omega}}) = h[0] + h[1]e^{-j\hat{\omega}} + h[2]e^{-j2\hat{\omega}} + h[3]e^{-j3\hat{\omega}} + \cdots$$
$$= \left|H(e^{j\hat{\omega}})\right| e^{j\angle H(e^{j\hat{\omega}})}$$

FIRST DIFFERENCE SYSTEM

Find $h[n]$ and $H(e^{j\hat{\omega}})$ for the Difference Equation: $\quad y[n] = x[n] - x[n-1]$

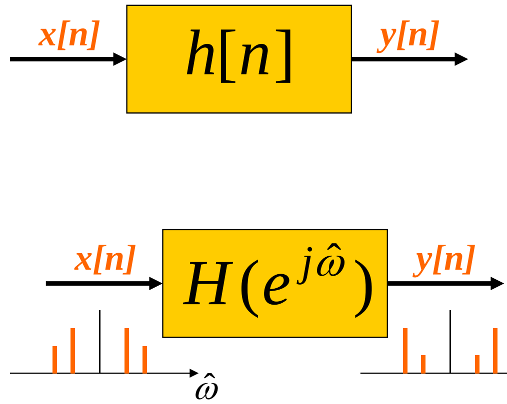


Ex: DELAY by 2 SYSTEM

Find $h[n]$ and $H(e^{j\hat{\omega}})$ for $y[n] = x[n-2]$

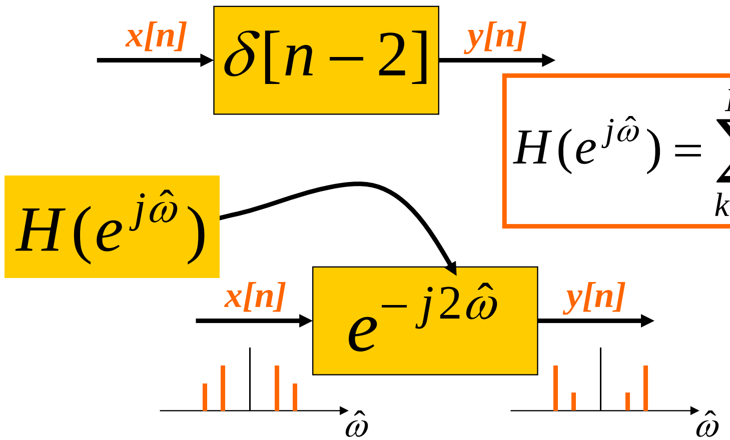
$b_k = \{\, 0, 0, 1 \,\}$

$h[n] = \delta[n-2]$



DELAY by 2 SYSTEM

Find $h[n]$ and $H(e^{j\hat{\omega}})$ for $y[n] = x[n-2]$



$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} \delta[k-2] e^{-j\hat{\omega}k}$$

k = 2 ONLY

GENERAL DELAY PROPERTY

Find $h[n]$ and $H(e^{j\hat{\omega}})$ for $y[n] = x[n-n_d]$

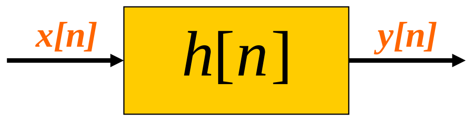
$h[n] = \delta[n-n_d]$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} \delta[k-n_d] e^{-j\hat{\omega}k} = e^{-j\hat{\omega}n_d}$$

ONLY ONE non-ZERO TERM for k at k = n_d

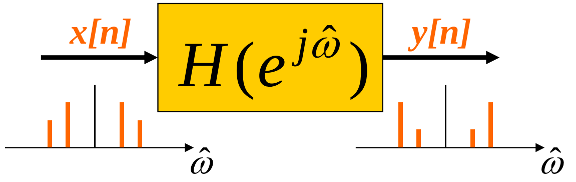
FREQ DOMAIN → TIME ??

- START with $H(e^{j\hat{\omega}})$ and find $h[n]$ or b_k



$h[n] = ?$

$$H(e^{j\hat{\omega}}) = 7e^{-j2\hat{\omega}}\cos(\hat{\omega})$$



FREQ DOMAIN --> TIME

$$H(e^{j\hat{\omega}}) = 7e^{-j2\hat{\omega}}\cos(\hat{\omega})$$

EULER's Formula

$$= 7e^{-j2\hat{\omega}}(0.5e^{j\hat{\omega}} + 0.5e^{-j\hat{\omega}})$$

$$= (3.5e^{-j\hat{\omega}} + 3.5e^{-j3\hat{\omega}})$$

$$h[n] = 3.5\delta[n-1] + 3.5\delta[n-3]$$

$$b_k = \{0, 3.5, 0, 3.5\}$$

PREVIOUS LECTURE REVIEW

- SINUSOIDAL INPUT SIGNAL
  - OUTPUT has SAME FREQUENCY
  - DIFFERENT Amplitude and Phase
- FREQUENCY RESPONSE of FIR
  - MAGNITUDE vs. Frequency
  - PHASE vs. Freq
  - PLOTTING

MAG **PHASE**

$$H(e^{j\hat{\omega}}) = |H(e^{j\hat{\omega}})| e^{j\angle H(e^{j\hat{\omega}})}$$

FREQ. RESPONSE PLOTS

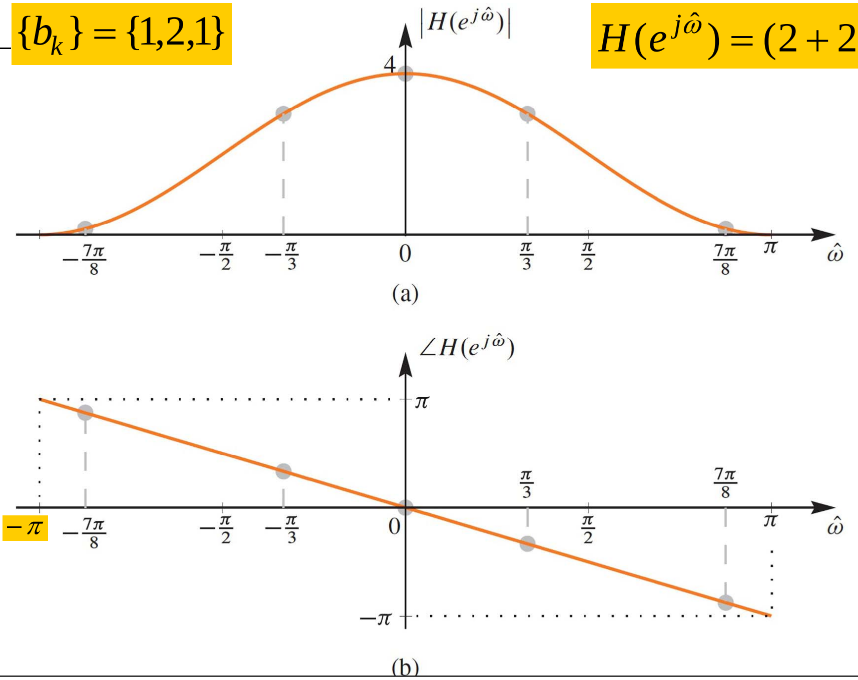
- DENSE GRID `(ww)` from $-\pi$ to $+\pi$
  - `ww = -pi:(pi/100):pi;`
- `HH = freqz(bb,1,ww)`
  - VECTOR `bb` contains Filter Coefficients
  - SP-First: `HH = freekz(bb,1,ww)`

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

PLOT of FREQ RESPONSE

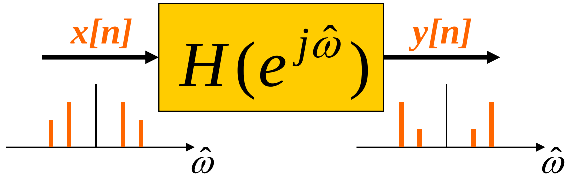
$\{b_k\} = \{1,2,1\}$

$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$



EXAMPLE 6.2

Find $y[n]$ when $H(e^{j\hat{\omega}})$ is known and $x[n] = 2e^{j\pi/4}e^{j(\pi/3)n}$

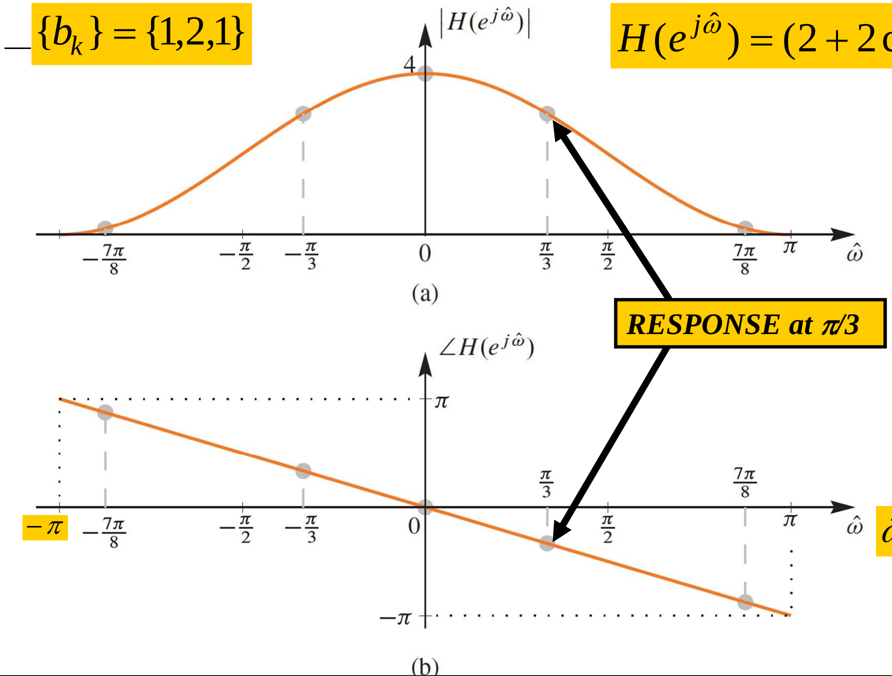


$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

PLOT of FREQ RESPONSE

$\{b_k\} = \{1,2,1\}$

$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$



EXAMPLE 6.2 (answer)

Find $y[n]$ when $x[n] = 2e^{j\pi/4}e^{j(\pi/3)n}$

One Step - evaluate $H(e^{j\hat{\omega}})$ at $\hat{\omega} = \pi/3$

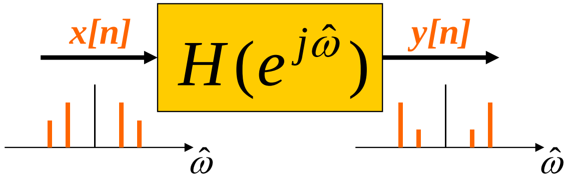
$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

$$H(e^{j\hat{\omega}}) = 3e^{-j\pi/3} \quad @\ \hat{\omega} = \pi/3$$

$$y[n] = \left(3e^{-j\pi/3}\right) \times 2e^{j\pi/4}e^{j(\pi/3)n} = 6e^{-j\pi/12}e^{j(\pi/3)n}$$

EXAMPLE: COSINE INPUT

Find $y[n]$ when $H(e^{j\hat{\omega}})$ is known and $x[n] = 2\cos(\frac{\pi}{3}n + \frac{\pi}{4})$



$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

EX: COSINE INPUT (ans-1)

Find $y[n]$ when $x[n] = 2\cos(\frac{\pi}{3}n + \frac{\pi}{4})$

$$2\cos(\tfrac{\pi}{3}n + \tfrac{\pi}{4}) = e^{j(\pi n/3 + \pi/4)} + e^{-j(\pi n/3 + \pi/4)}$$
$$\Rightarrow x[n] = x_1[n] + x_2[n]$$

$$y_1[n] = H(e^{j\pi/3})e^{j(\pi n/3 + \pi/4)}$$
$$y_2[n] = H(e^{-j\pi/3})e^{-j(\pi n/3 + \pi/4)}$$
$$\Rightarrow y[n] = y_1[n] + y_2[n]$$

EX: COSINE INPUT (ans-2)

Find $y[n]$ when $x[n] = 2\cos(\frac{\pi}{3}n + \frac{\pi}{4})$

$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

$$y_1[n] = H(e^{j\pi/3})e^{j(\pi n/3 + \pi/4)} = 3e^{-j(\pi/3)}e^{j(\pi n/3 + \pi/4)}$$
$$y_2[n] = H(e^{-j\pi/3})e^{-j(\pi n/3 + \pi/4)} = 3e^{j(\pi/3)}e^{-j(\pi n/3 + \pi/4)}$$

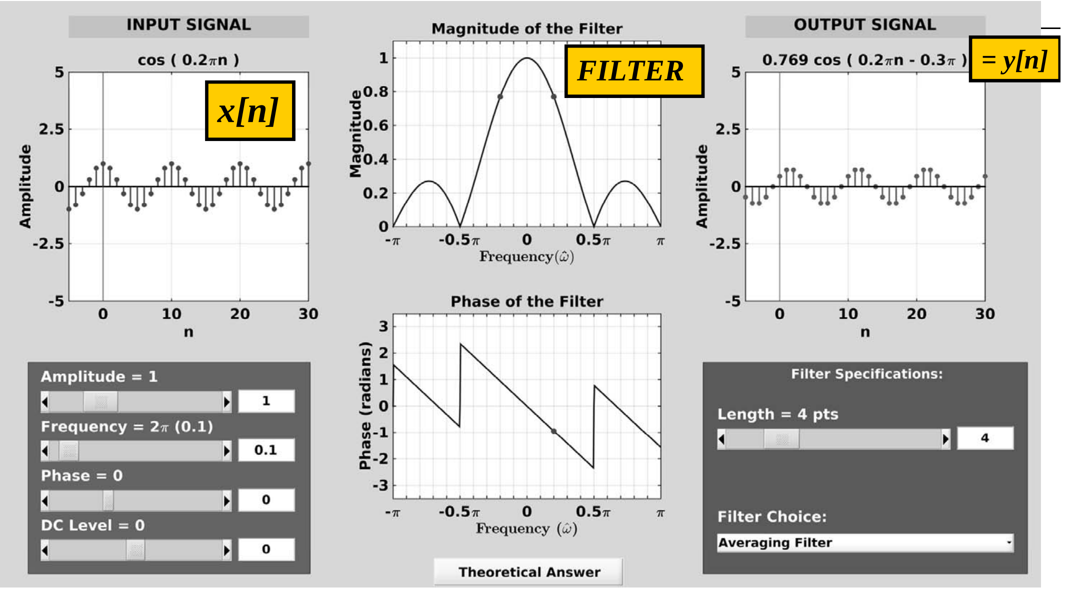
$$y[n] = 3e^{j(\pi n/3 - \pi/12)} + 3e^{-j(\pi n/3 - \pi/12)}$$
$$\Rightarrow y[n] = 6\cos(\tfrac{\pi}{3}n - \tfrac{\pi}{12})$$

SINUSOID thru FIR

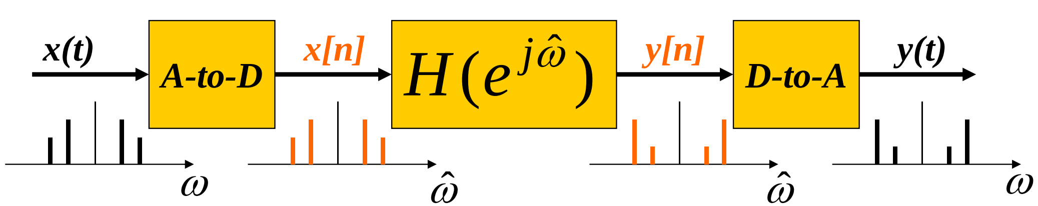
- IF $H^*(e^{j\hat{\omega}}) = H(e^{-j\hat{\omega}})$ ← *When $\mathbf{b_k}$'s are real valued*
- Multiply the Magnitudes
- Add the Phases

$$x[n] = A\cos(\hat{\omega}_1 n + \phi)$$
$$\Rightarrow y[n] = A\left|H(e^{j\hat{\omega}_1})\right|\cos(\hat{\omega}_1 n + \phi + \angle H(e^{j\hat{\omega}_1}))$$

DLTI Demo with Sinusoids



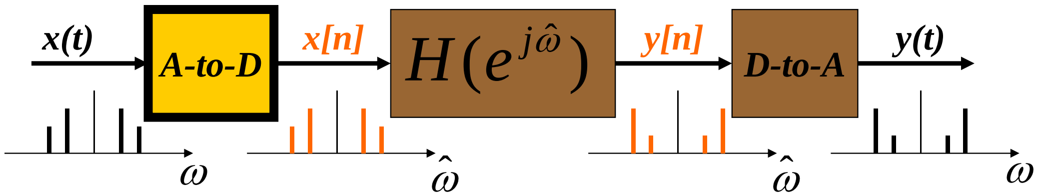
DIGITAL "FILTERING"



- ω SPECTRUM of x(t) (SUM of SINUSOIDS)
- $\hat{\omega}$ SPECTRUM of x[n] (ALIASING a PROBLEM?)
- SPECTRUM y[n] (FIR Gain or Nulls)
- ω Then, OUTPUT y(t) = SUM of SINUSOIDS

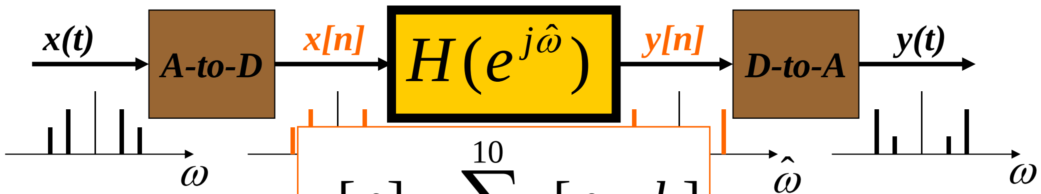
Question: how to characterize the effects of a digital filter on the analog signals? Answer: Frequency Scaling

FREQUENCY SCALING



- TIME SAMPLING: $t = nT_s$
  - IF NO ALIASING:
  - FREQUENCY SCALING $\hat{\omega} = \omega T_s = \dfrac{\omega}{f_s}$

11-pt Running Sum Example



$$y[n] = \sum_{k=0}^{10} x[n-k]$$

250 Hz

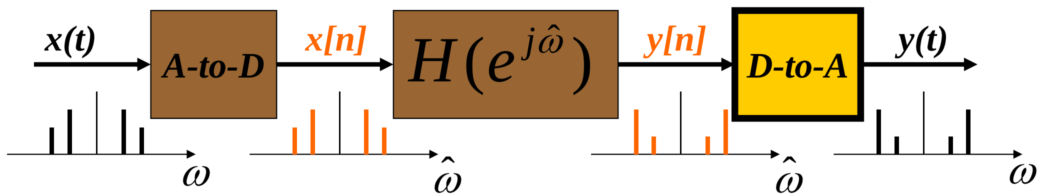
25 Hz

$$H(e^{j\hat{\omega}}) = \frac{\sin(\frac{11}{2}\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})} e^{-j5\hat{\omega}}$$

?

$$x(t) = \cos(2\pi(25)t) + \cos(2\pi(250)t - \tfrac{1}{2}\pi)$$

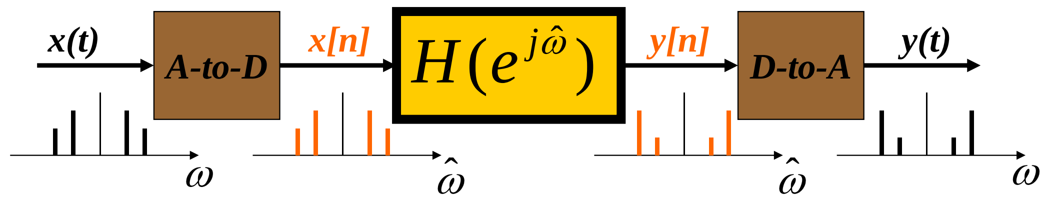
D-A FREQUENCY SCALING



- *TIME SAMPLING:* $t = nT_s \Rightarrow n \leftarrow t f_s$
- RECONSTRUCT up to $0.5f_s$
  - FREQUENCY SCALING

$$\omega = \hat{\omega} f_s$$

TRACK the FREQUENCIES

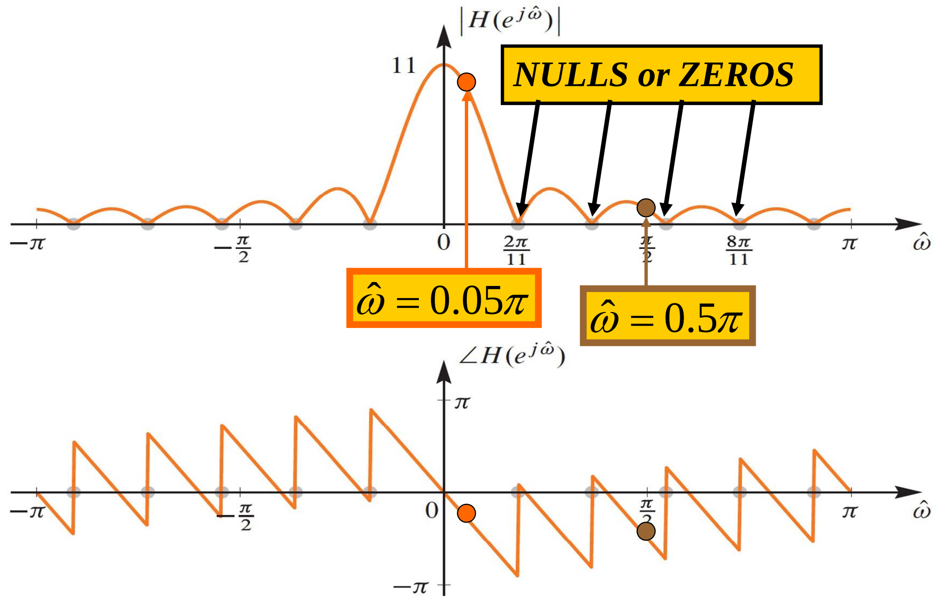


250 Hz	0.5π	$H(e^{j0.5\pi})$	0.5π	250 Hz
25 Hz	$.05\pi$	$H(e^{j0.05\pi})$	$.05\pi$	25 Hz

Fs = 1000 Hz

NO new freqs

11-pt Running Sum



(b)

EVALUATE Freq. Response

$$H(e^{j\hat{\omega}}) = \frac{\sin(\frac{11}{2}\hat{\omega})}{11\sin(\frac{1}{2}\hat{\omega})} e^{-j5\hat{\omega}}$$

At $\hat{\omega} = 0.5\pi$

$$H(e^{j\hat{\omega}}) = \frac{\sin(\frac{11}{2}(0.5\pi))}{11\sin(\frac{1}{2}(0.5\pi))} e^{-j5(0.5\pi)}$$

$$= \frac{\sin(2.75\pi)}{11\sin(0.25\pi)} e^{-j2.5\pi}$$

$$= 0.0909 e^{-j0.5\pi}$$

EVALUATE Freq. Response

$$x(t) = \cos(2\pi(25)t) + \sin(2\pi(250)t)$$

evaluating at 25 and 250 Hz.

$f_s = 1000$

$$H(e^{j2\pi(25)/1000}) = \frac{\sin(\pi(25)(11)/1000)}{11\sin(\pi(25)/1000)} e^{-j2\pi(25)(5)/1000}$$

$$= 0.8811 e^{-j\pi/4}$$

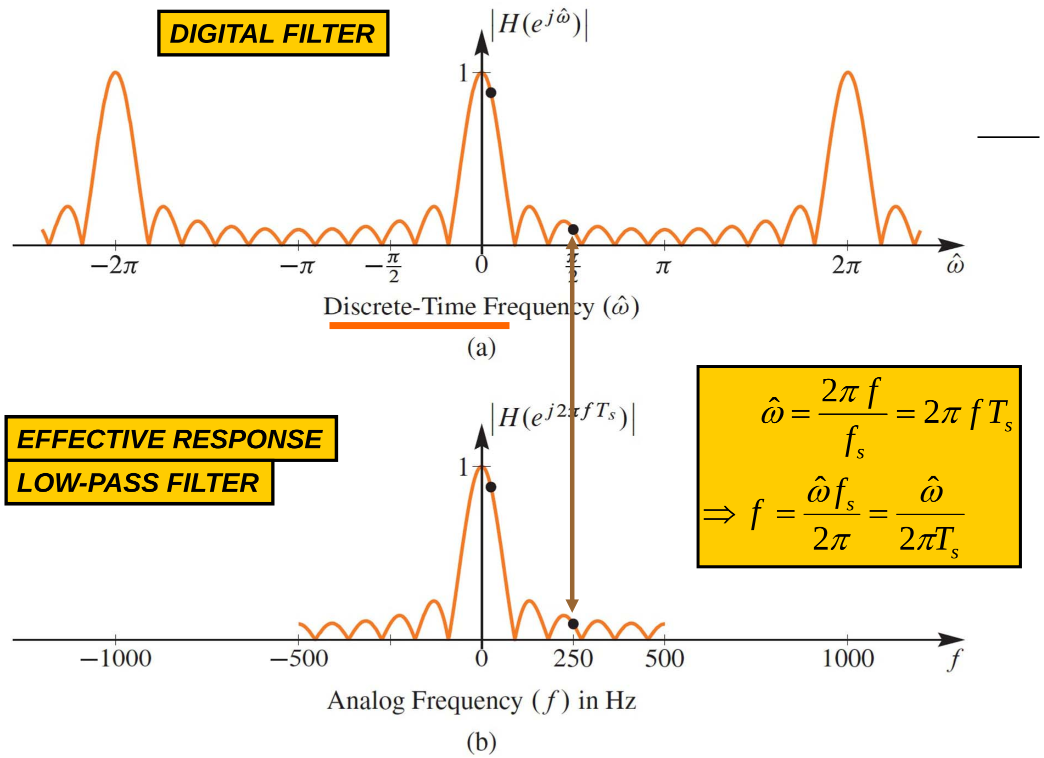
MAG SCALE

PHASE CHANGE

$$H(e^{j2\pi(250)/1000}) = \frac{\sin(\pi(250)(11)/1000)}{11\sin(\pi(250)/1000)} e^{-j2\pi(250)(5)/1000}$$

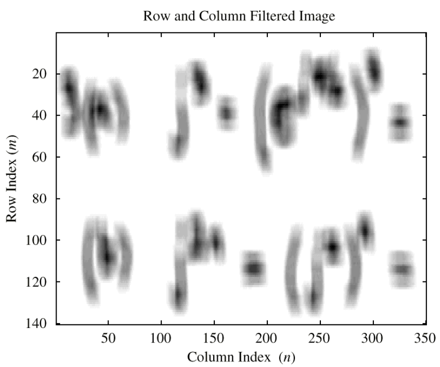
$$= 0.0909 e^{-j\pi/2}$$

$$y(t) = 0.8811\cos(2\pi(25)t - \pi/4) + 0.0909\sin(2\pi(250)t - \pi/2)$$

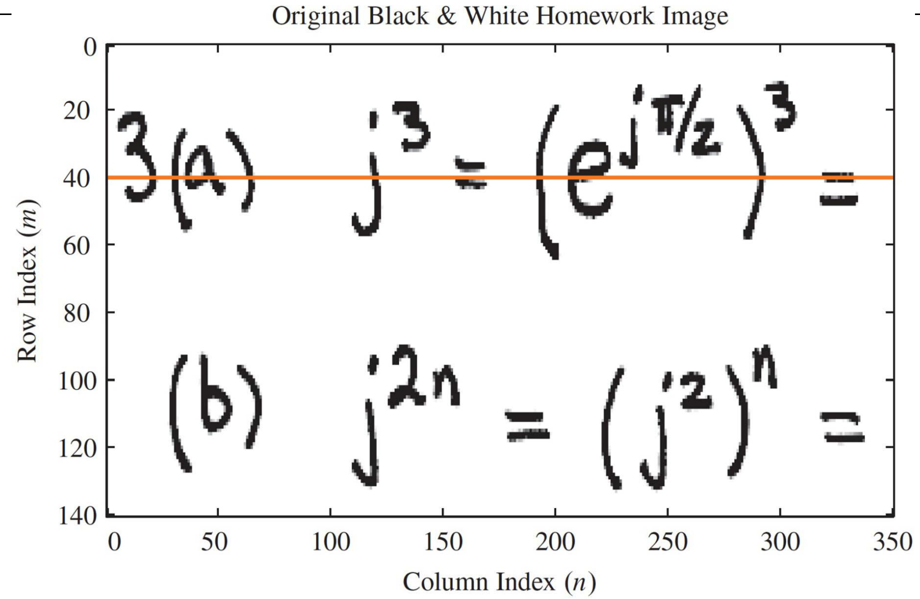


FILTER TYPES

- LOW-PASS FILTER (LPF)
  - BLURRING
  - ATTENUATES HIGH FREQUENCIES
- HIGH-PASS FILTER (HPF)
  - SHARPENING for IMAGES
  - BOOSTS THE HIGHS
  - REMOVES DC
- BAND-PASS FILTER (BPF)

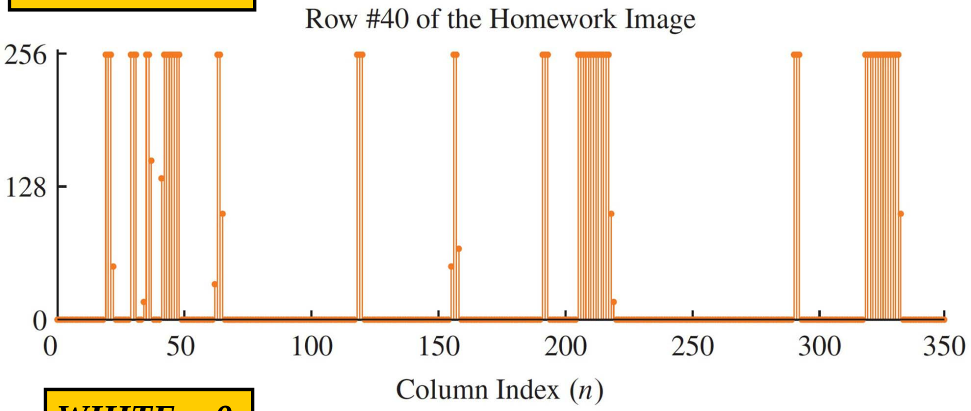


B & W IMAGE



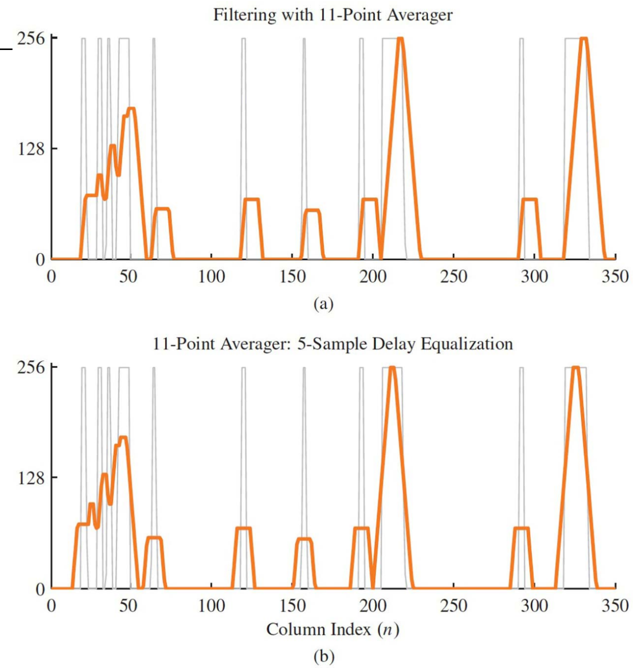
ROW of B&W IMAGE

BLACK = 255



WHITE = 0

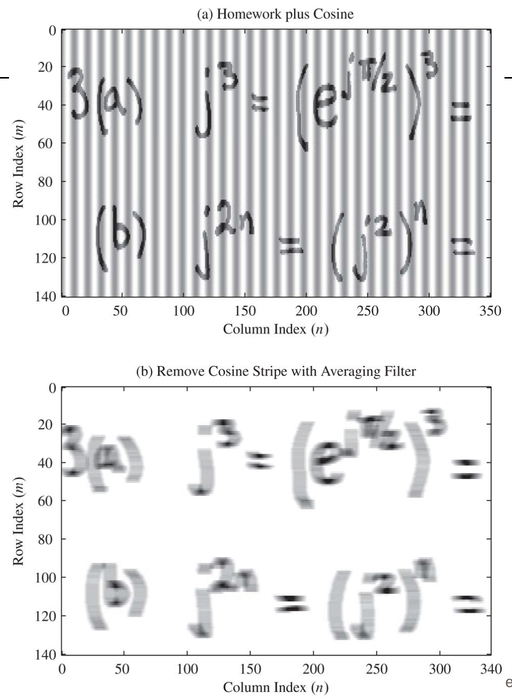
FILTERED ROW of IMAGE



Original is gray
Filtered is orange

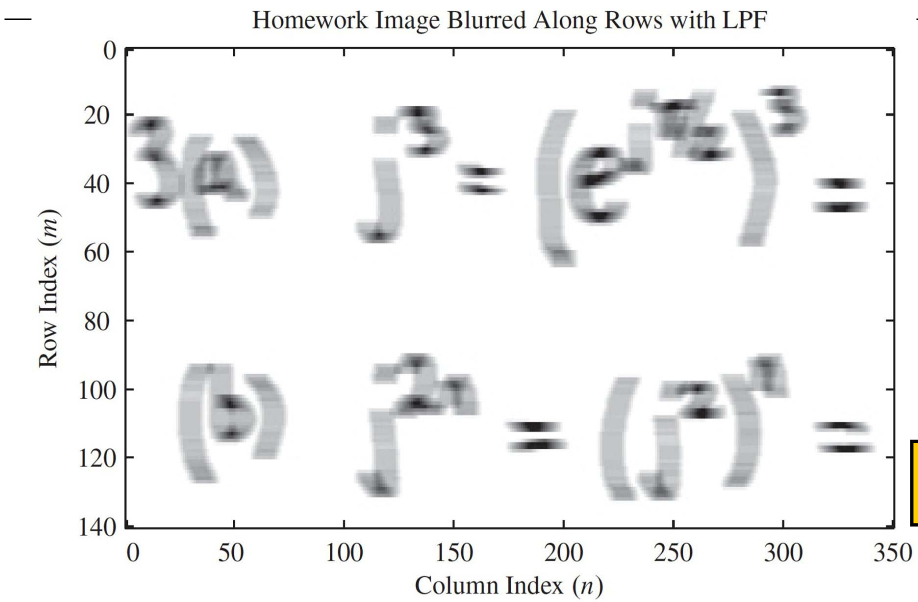
DELAY adjusted by 5 samples

IMAGE with COSINE ADDED



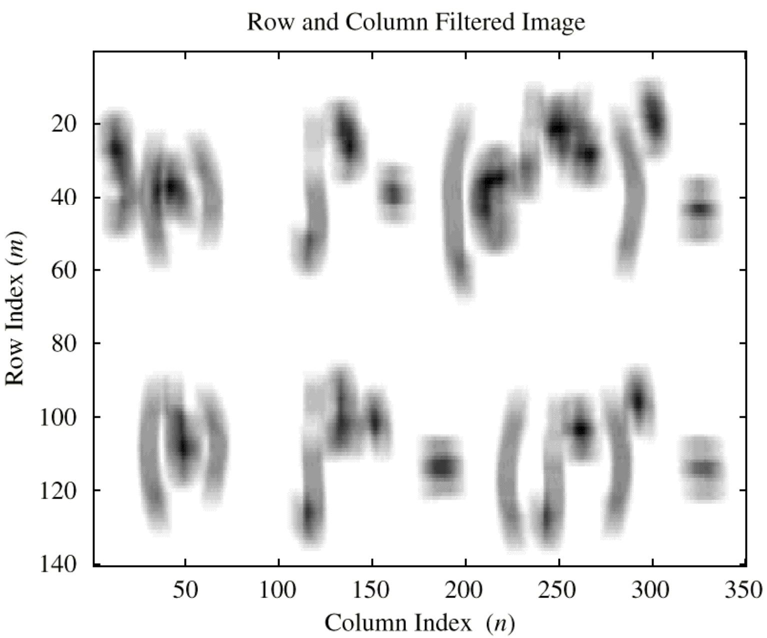
FILTERED
with 11-pt running average
across horizontal dimension
→ yields horizontal blur

FILTERED B&W IMAGE



Horizontal Blur only

FILTER ROWS & COLUMNS



Blur in both dimensions