DSP First, 2/e

Lecture 15

DTFT: Discrete-Time Fourier Transform

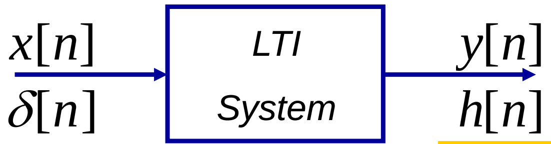
READING ASSIGNMENTS

- This Lecture:
  - Chapter 7, Section 7-1

Lecture Objective

- Generalize the Frequency Response
- Introduce **DTFT**, discrete-time Fourier transform, for discrete time sequences that may not be finite or periodic
- Establish general concept of "**frequency** **domain**" representations and **spectrum** that is a **continuous** function of (normalized) frequency – not necessarily just a line spectrum

The Frequency Response



$e^{j\hat{\omega}_0 n}$ → $H(e^{j\hat{\omega}_0})e^{j\hat{\omega}_0 n}$

$$H(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\hat{\omega}n}$$

$$\text{Periodic}: H(e^{j(\hat{\omega}+2\pi)}) = H(e^{j\hat{\omega}})$$

$$y[n] = A \cdot \left|H(e^{j\hat{\omega}_0})\right| \cos(\hat{\omega}_0 n + \varphi + \angle H(e^{j\hat{\omega}_0}))$$

Discrete-Time Fourier Transform

- Definition of the **DTFT**: $X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$
- Forward DTFT $x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}d\hat{\omega}$
- **Inverse DTFT** $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}d\hat{\omega}$
- Always periodic with a period of 2π

$$X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}})$$

What is a Transform?

- Change problem from one domain to another to make it easier
- Example: Phasors
  - Solve simultaneous sinusoid equations (hard)
  - Invert a matrix of phasors (easy)
- Has to be invertible
  - Transform into new domain
  - Return to original domain (inverse must be unique)

Periodicity of DTFT

- For any integer m: $X(e^{j\hat{\omega}}) = X(e^{j(\hat{\omega}+2m\pi)})$

$$X(e^{j(\hat{\omega}+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\hat{\omega}+2m\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}e^{-j2\pi mn} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

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Existence of DTFT

- Discrete-time Fourier transform (DTFT) exists – provided that the sequence is **absolutely-summable**

$$\left|X(e^{j\hat{\omega}})\right| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}\right|$$

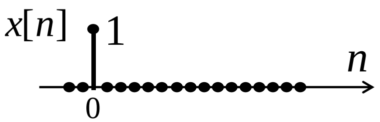
$$\leq \sum_{n=-\infty}^{\infty} \left|x[n]e^{-j\hat{\omega}n}\right| = \sum_{n=-\infty}^{\infty} \left|x[n]\right| < \infty$$

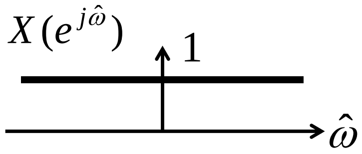
- DTFT applies to discrete time sequences, $x[n]$, regardless of length (if $x[n]$ is **absolute summable**)

DTFT of a Single Sample

$$x[n] = \begin{cases} 1, & n=0 \\ 0, & \text{elsewhere} \end{cases} = \delta[n]$$

Unit Impulse function





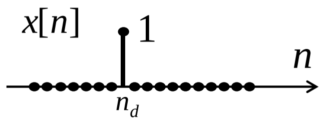
$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\hat{\omega}n}$$

$$= \sum_{n=0}^{0} e^{-j\hat{\omega}n} = 1$$

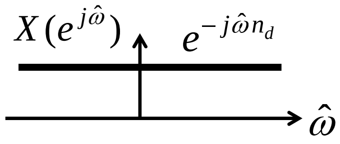
$$x[n] = \delta[n] \overset{\mathscr{F}}{\Leftrightarrow} X(e^{j\hat{\omega}}) = 1$$

Delayed Unit Impulse

$$x_d[n] = \delta[n-n_d] = \begin{cases} 1, & n=n_d \\ 0, & \text{elsewhere} \end{cases}$$



$$X_d(e^{j\hat{\omega}}) = \sum_{n=n_d}^{n_d} e^{-j\hat{\omega}n} = e^{-j\hat{\omega}n_d}$$



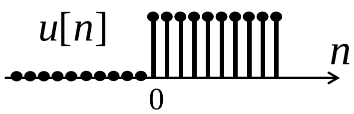
Generalizes to the <u>delay property</u>

$$x_d[n] = x[n-n_d] \Leftrightarrow$$

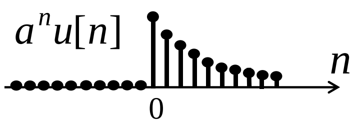
$$X_d(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}}) e^{-j\hat{\omega}n_d} = e^{-j\hat{\omega}n_d}$$

DTFT of Right-Sided Exponential

Unit Step Function: $u[n] = \begin{cases} 0, & n<0 \\ 1, & n \geq 0 \end{cases}$



$$x[n] = a^n u[n], \quad |a| < 1$$



$$X(e^{j\hat{\omega}}) = \sum_{n=0}^{\infty} a^n e^{-j\hat{\omega}n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\hat{\omega}})^n = \frac{1}{1-ae^{-j\hat{\omega}}} \quad \text{if } |a| < 1$$

Plotting: Magnitude and Angle Form

$$x[n] = a^n u[n] \Leftrightarrow X(e^{j\hat{\omega}}) = \frac{1}{1-ae^{-j\hat{\omega}}}$$

$$X(e^{j\hat{\omega}}) = |X(e^{j\hat{\omega}})| e^{j\angle X(e^{j\hat{\omega}})}$$

$$|X(e^{j\hat{\omega}})|^2 = X(e^{j\hat{\omega}}) X^*(e^{j\hat{\omega}}) = \frac{1}{1-ae^{-j\hat{\omega}}} \cdot \frac{1}{1-ae^{j\hat{\omega}}}$$

$$= \frac{1}{1+a^2-2a\cos(\hat{\omega})}$$

$$\angle X(e^{j\hat{\omega}}) = \arctan\left(\frac{-a\sin(\hat{\omega})}{1-a\cos(\hat{\omega})}\right)$$

Magnitude and Angle Plots

EVEN Function

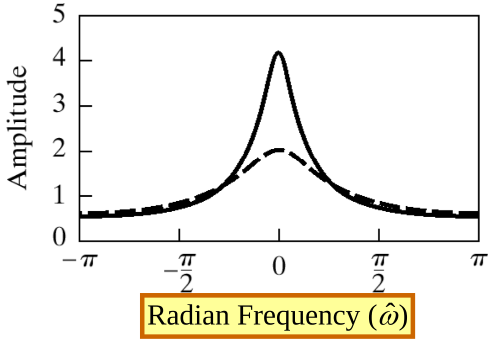
$$\left|X(e^{j\hat{\omega}})\right| = \frac{1}{(1+a^2-2a\cos(\hat{\omega}))^{1/2}}$$

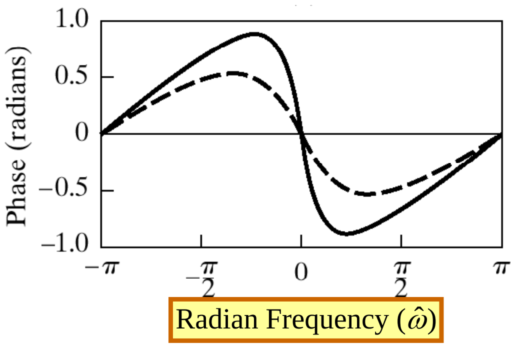
$$\left|X(e^{-j\hat{\omega}})\right| = \left|X(e^{j\hat{\omega}})\right|$$

ODD Function

$$\angle X(e^{j\hat{\omega}}) = \arctan\left(\frac{-a\sin(\hat{\omega})}{1-a\cos(\hat{\omega})}\right)$$

$$\angle X(e^{-j\hat{\omega}}) = -\angle X(e^{j\hat{\omega}})$$





Inverse DTFT ?

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}\,d\hat{\omega}$$

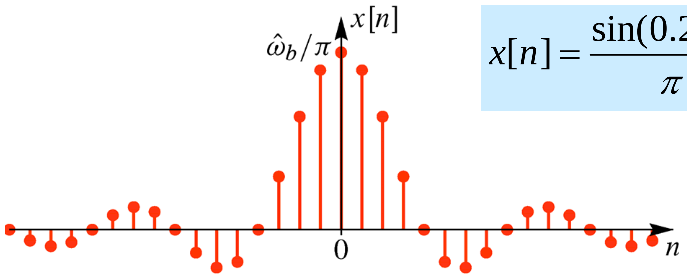
$$X(e^{j\hat{\omega}}) = \frac{1}{1+0.3e^{-j\hat{\omega}}} \quad \Rightarrow x[n] = ?$$

$$x[n] = \int_{-\pi}^{\pi} \frac{1}{1+0.3e^{-j\hat{\omega}}} e^{j\hat{\omega}n}\,d\hat{\omega} \quad ??$$

$$x[n] = a^n u[n] \Leftrightarrow X(e^{j\hat{\omega}}) = \frac{1}{1-ae^{-j\hat{\omega}}}$$

SINC Function:

- A "**sinc**" function or sequence



$$x[n] = \frac{\sin(0.25\pi n)}{\pi n}, \quad -\infty < n < \infty$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} \frac{\sin(0.25\pi n)}{\pi n} e^{-j\hat{\omega}n} = \quad ??$$

SINC Function from the inverse DTFT integral

Given a "**sinc**" function or sequence

$$x[n] = \frac{\sin(0.2\pi n)}{\pi n}, \quad -\infty < n < \infty$$

Consider an ideal band-limited signal:

$$X(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \le 0.2\pi \\ 0, & 0.2\pi < |\hat{\omega}| \le \pi \end{cases}$$

Discrete-time Fourier Transform Pair

$$x[n] = \frac{1}{2\pi}\int_{-0.2\pi}^{0.2\pi} e^{j\hat{\omega}n}\,d\hat{\omega} = \left.\frac{e^{j\hat{\omega}n}}{2\pi\, jn}\right|_{-0.2\pi}^{0.2\pi}$$

$$= \frac{e^{j0.2\pi n} - e^{-j0.2\pi n}}{2\pi\, jn} = \frac{\sin(0.2\pi n)}{\pi n}$$

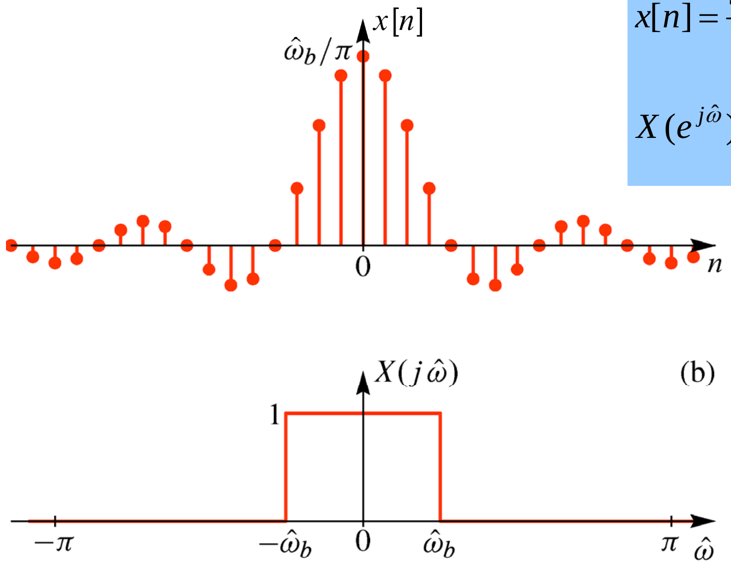
$$x[n] = \frac{\sin(\hat{\omega}_b n)}{\pi n} \overset{DTFT}{\Leftrightarrow}$$

$$X(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \le \hat{\omega}_b \\ 0, & \hat{\omega}_b < |\hat{\omega}| \le \pi \end{cases}$$

SINC Function – Rectangle DTFT pair

$$x[n]=\frac{\sin(\hat{\omega}_b n)}{\pi n} \overset{DTFT}{\Leftrightarrow}$$

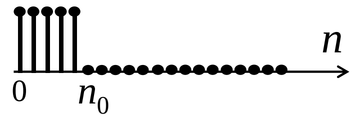
$$X(e^{j\hat{\omega}})=\begin{cases}1, & |\hat{\omega}|\le\hat{\omega}_b\\ 0, & \hat{\omega}_b<|\hat{\omega}|\le\pi\end{cases}$$



DTFT of Rectangular Pulse

A "**rectangular**" sequence of length *L*

$$x[n]=\begin{cases}1, & 0\le n\le L-1\\ 0, & \text{elsewhere}\end{cases}$$



$$X(e^{j\hat{\omega}})=\sum_{n=0}^{L-1}e^{-j\hat{\omega}n}=\frac{1-e^{-jL\hat{\omega}}}{1-e^{-j\hat{\omega}}}$$

$$=\frac{e^{-j(L-1)\hat{\omega}/2}\left(\sin\frac{L\hat{\omega}}{2}\right)}{\left(\sin\frac{\hat{\omega}}{2}\right)}$$

Discrete-time Fourier Transform Pair

Dirichlet Function:

$D_L(e^{j\hat{\omega}})$

Summary of DTFT Pairs

$$x[n]=\delta[n-n_d]\Leftrightarrow X(e^{j\hat{\omega}})=e^{-j\hat{\omega}n_d}$$

Delayed Impulse

$$x[n]=a^n u[n]\Leftrightarrow X(e^{j\hat{\omega}})=\frac{1}{1-ae^{-j\hat{\omega}}}$$

Right-sided Exponential

$$x[n]=\frac{\sin(\hat{\omega}_c n)}{\pi n}\Leftrightarrow X(e^{j\hat{\omega}})=\begin{cases}1 & |\hat{\omega}|\le\hat{\omega}_c\\ 0 & \hat{\omega}_c<|\hat{\omega}|<\pi\end{cases}$$

sinc function is ***Bandlimited***

$$x[n]=\begin{cases}1 & 0\le n<L\\ 0 & \text{otherwise}\end{cases}\Leftrightarrow X(e^{j\hat{\omega}})=\frac{\sin(\frac{1}{2}L\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}e^{-j\hat{\omega}(L-1)/2}$$

Using the DTFT

- The DTFT provides a ***frequency-domain*** representation that is invaluable for thinking about signals and solving DSP problems.
- To use it effectively you must
  - know **PAIRS**: the Fourier transforms of certain important signals
  - know ***properties*** and certain key ***theorems***
  - be able to combine time-domain and frequency domain methods appropriately

Summary

Discrete-time Fourier Transform (DTFT)

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\hat{\omega} n}$$

Inverse Discrete-time Fourier Transform

$$x[n] = \frac{1}{2\pi} \int_{0}^{2\pi} X(e^{j\hat{\omega}}) e^{j\hat{\omega} n} d\hat{\omega}$$