DSP First, 2/e

Lecture 16

DTFT Properties

READING ASSIGNMENTS

- This Lecture:
  - Chapter 7, Sections 7-2 & 7-3
- Other Reading:
  - Recitation: Chapter 7
    - DTFT EXAMPLES

Lecture Objectives

- Properties of the DTFT
- **Convolution is mapped to Multiplication**
- Ideal Filters: LPF, HPF, & BPF
- Recall:
  - DTFT is the math behind the general concept of "**frequency domain**" representations
  - The **spectrum** is now a **continuous** function of (normalized) frequency – not just a line spectrum

Discrete-Time Fourier Transform

- Definition of the **DTFT**: $X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$
- Forward DTFT $x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}d\hat{\omega}$
- **Inverse DTFT** $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}d\hat{\omega}$
- Always periodic with a period of 2π

$$X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}})$$

Summary of DTFT Pairs

$$x[n]=\delta[n-n_d]\Leftrightarrow X(e^{j\hat{\omega}})=e^{-j\hat{\omega}n_d}$$

Delayed Impulse

$$x[n]=a^n u[n]\Leftrightarrow X(e^{j\hat{\omega}})=\frac{1}{1-ae^{-j\hat{\omega}}}$$

Right-sided Exponential

$$x[n]=\frac{\sin(\hat{\omega}_c n)}{\pi n}\Leftrightarrow X(e^{j\hat{\omega}})=\begin{cases}1 & |\hat{\omega}|\le\hat{\omega}_c\\ 0 & \hat{\omega}_c<|\hat{\omega}|<\pi\end{cases}$$

*sinc function is **Bandlimited***

$$x[n]=\begin{cases}1 & 0\le n<L\\ 0 & \text{otherwise}\end{cases}\Leftrightarrow X(e^{j\hat{\omega}})=\frac{\sin(\frac{1}{2}L\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}e^{-j\hat{\omega}(L-1)/2}$$

Conjugate Symmetry

$$x[n]=x^*[n]\quad(\text{real-valued})$$
$$\Rightarrow X(e^{j\hat{\omega}})=X^*(e^{-j\hat{\omega}})$$

$$y[n]=x^*[n]$$

$$Y(e^{j\hat{\omega}})=\sum_{n=-\infty}^{\infty}x^*[n]e^{-j\hat{\omega}n}$$

$$=\left(\sum_{n=-\infty}^{\infty}x[n]e^{-j(-\hat{\omega})n}\right)^*=X^*(e^{-j\hat{\omega}})$$

$$x[n]=a^n u[n]\Leftrightarrow X(e^{j\hat{\omega}})=\frac{1}{1-ae^{-j\hat{\omega}}}$$

Magnitude and Angle Plots

EVEN Function

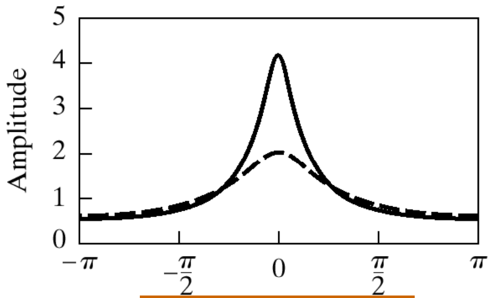
$$|X(e^{j\hat{\omega}})|=\frac{1}{(1+a^2-2a\cos(\hat{\omega}))^{1/2}}$$

$$|X(e^{-j\hat{\omega}})|=|X(e^{j\hat{\omega}})|$$

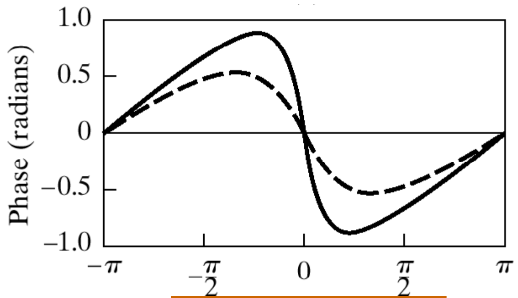
ODD Function

$$\angle X(e^{j\hat{\omega}})=\arctan\left(\frac{-a\sin(\hat{\omega})}{1-a\cos(\hat{\omega})}\right)$$

$$\angle X(e^{-j\hat{\omega}})=-\angle X(e^{j\hat{\omega}})$$



Radian Frequency ($\hat{\omega}$)



Radian Frequency ($\hat{\omega}$)

Properties of DTFT

- Linearity

$$x[n]=ax_1[n]+bx_2[n]\Leftrightarrow X(e^{j\hat{\omega}})=aX_1(e^{j\hat{\omega}})+bX_2(e^{j\hat{\omega}})$$

- Time-Delay ←→ phase shift

$$y[n]=x[n-n_d]\Leftrightarrow Y(e^{j\hat{\omega}})=X(e^{j\hat{\omega}})e^{-j\hat{\omega}n_d}$$

- Frequency-Shift ←→ multiply by sinusoid

$$y[n]=e^{j\hat{\omega}_c n}x[n]\Leftrightarrow Y(e^{j\hat{\omega}})=X(e^{j(\hat{\omega}-\hat{\omega}_c)})$$

Linearity (Proof)

$$x[n] = ax_1[n] + bx_2[n] \Leftrightarrow X(e^{j\hat{\omega}}) = aX_1(e^{j\hat{\omega}}) + bX_2(e^{j\hat{\omega}})$$

$$X_1(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x_1[n]e^{-j\hat{\omega}n} \qquad X_2(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x_2[n]e^{-j\hat{\omega}n}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} (ax_1[n] + bx_2[n])e^{-j\hat{\omega}n}$$

$$= \sum_{n=-\infty}^{\infty} (ax_1[n])e^{-j\hat{\omega}n} + \sum_{n=-\infty}^{\infty} (bx_2[n])e^{-j\hat{\omega}n}$$

$$= aX_1(e^{j\hat{\omega}}) + bX_2(e^{j\hat{\omega}})$$

Time-Delay Property (Proof)

$$y[n] = x[n - n_d] \Leftrightarrow Y(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}})e^{-j\hat{\omega}n_d}$$

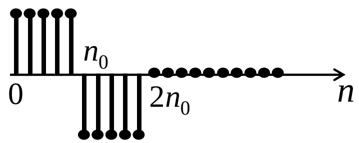
$$Y(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n - n_d]e^{-j\hat{\omega}n}$$

$$= e^{-j\hat{\omega}n_d} \sum_{n=-\infty}^{\infty} x[n - n_d]e^{-j\hat{\omega}(n - n_d)}$$

$$= e^{-j\hat{\omega}n_d} \sum_{m=-\infty}^{\infty} x[m]e^{-j\hat{\omega}(m)} = X(e^{j\hat{\omega}})e^{-j\hat{\omega}n_d}$$

Use Properties to Find DTFT

$$y[n] = \begin{cases} 1, & 0 \le n \le n_0 - 1 \\ -1, & n_0 \le n \le 2n_0 - 1 \\ 0, & \text{elsewhere} \end{cases}$$



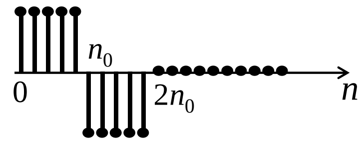
$$y[n] = x[n] - x[n - n_0]$$

Strategy: Exploit Known Transform Pair

$$x[n] = \begin{cases} 1, & 0 \le n \le n_0 - 1 \\ 0, & \text{elsewhere} \end{cases} \Rightarrow X(e^{j\hat{\omega}}) = \frac{e^{-j(n_0-1)\hat{\omega}/2}\left(\sin\frac{n_0\hat{\omega}}{2}\right)}{\left(\sin\frac{\hat{\omega}}{2}\right)}$$

Use Properties to Find DTFT (2)

$$y[n] = \begin{cases} 1, & 0 \le n \le n_0 - 1 \\ -1, & n_0 \le n \le 2n_0 - 1 \\ 0, & \text{elsewhere} \end{cases}$$

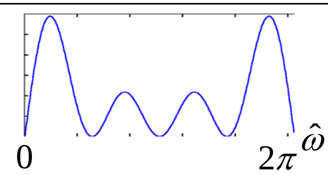


$$y[n] = x[n] - x[n - n_0] \quad \Leftrightarrow \quad Y(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}}) - e^{-j\hat{\omega}n_0}X(e^{j\hat{\omega}})$$

Strategy: Exploit Known Transform Pair

$$x[n] = \begin{cases} 1, & 0 \le n \le n_0 - 1 \\ 0, & \text{elsewhere} \end{cases} \Rightarrow X(e^{j\hat{\omega}}) = \frac{e^{-j(n_0-1)\hat{\omega}/2}\left(\sin\frac{n_0\hat{\omega}}{2}\right)}{\left(\sin\frac{\hat{\omega}}{2}\right)}$$

$$Y(e^{j\hat{\omega}}) = \frac{e^{-j(n_0-1)\hat{\omega}/2}\left(\sin\frac{n_0\hat{\omega}}{2}\right)\left(1 - e^{-jn_0\hat{\omega}}\right)}{\left(\sin\frac{\hat{\omega}}{2}\right)}$$



Frequency Shift

$$y[n] = e^{j\hat{\omega}_c n}x[n] \Leftrightarrow Y(e^{j\hat{\omega}}) = X(e^{j(\hat{\omega}-\hat{\omega}_c)})$$

$$Y(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} e^{j\hat{\omega}_c n}x[n]e^{-j\hat{\omega}n}$$

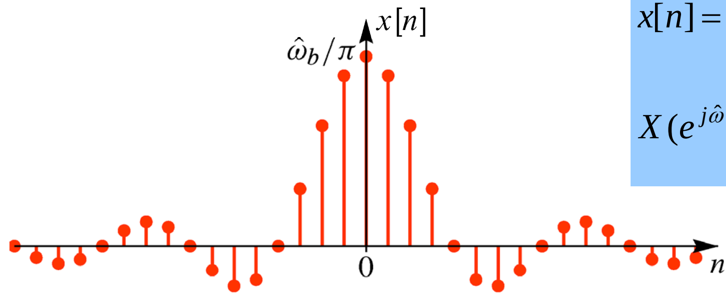
$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\hat{\omega}-\hat{\omega}_c)n}$$

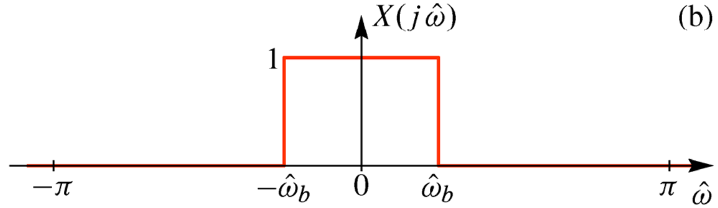
$$= X(e^{j(\hat{\omega}-\hat{\omega}_c)})$$

SINC Function – Rectangle DTFT pair

$$x[n] = \frac{\sin(\hat{\omega}_b n)}{\pi n} \overset{DTFT}{\Leftrightarrow}$$

$$X(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \le \hat{\omega}_b \\ 0, & \hat{\omega}_b < |\hat{\omega}| \le \pi \end{cases}$$





Sinc times sinusoid: find DTFT

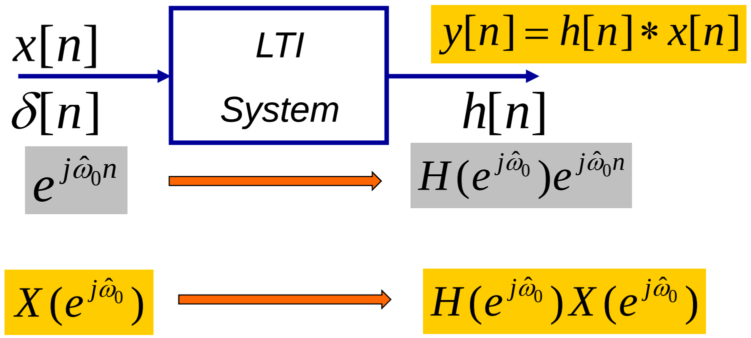
$$y[n] = \frac{\sin(\hat{\omega}_b n)}{\pi n} e^{j\hat{\omega}_a n} \quad \Leftrightarrow \quad Y(e^{j\hat{\omega}}) = ?$$

$$y_2[n] = \frac{\sin(\hat{\omega}_b n)}{\pi n} \underbrace{\cos(\hat{\omega}_a n)}_{\frac{1}{2}e^{j\hat{\omega}_a n} + \frac{1}{2}e^{-j\hat{\omega}_a n}} \quad \Leftrightarrow \quad Y_2(e^{j\hat{\omega}}) = ?$$

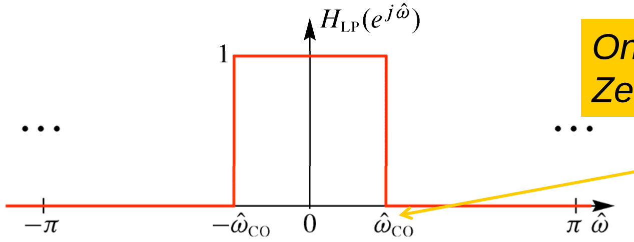
*Frequency shifting **up and down** is done by cosine multiplication in the time domain*

DTFT maps Convolution to Multiplication

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \quad \Leftrightarrow \quad Y(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}})X(e^{j\hat{\omega}})$$

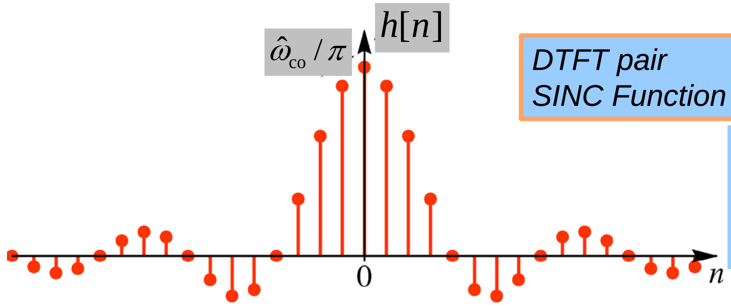


IDEAL LowPass Filter (LPF)



One in the passband
Zero in the stopband

$\hat{\omega}_{co}$ is cutoff frequency

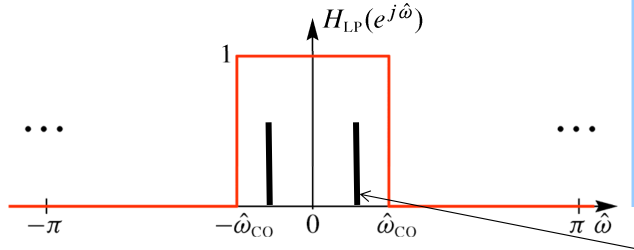


DTFT pair
SINC Function – Rectangle

$$h[n] = \frac{\sin(\hat{\omega}_{co} n)}{\pi n} \quad \overset{DTFT}{\Leftrightarrow}$$

$$H(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \le \hat{\omega}_{co} \\ 0, & \hat{\omega}_{co} < |\hat{\omega}| \le \pi \end{cases}$$

Filtering with the IDEAL LPF



$$h[n] = \frac{\sin(\hat{\omega}_{co} n)}{\pi n} \quad \overset{DTFT}{\Leftrightarrow}$$

$$H(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \le \hat{\omega}_{co} \\ 0, & \hat{\omega}_{co} < |\hat{\omega}| \le \pi \end{cases}$$

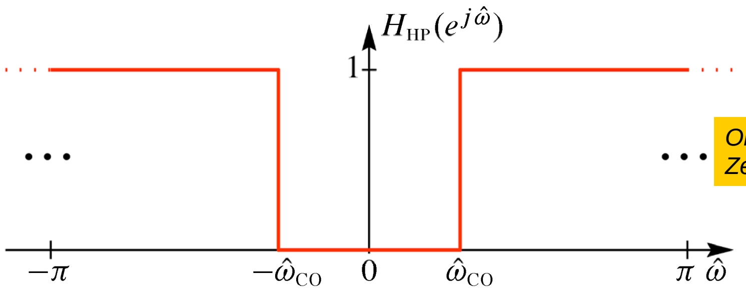
Find the output when the input is a sinusoid: $x[n] = \cos(\hat{\omega}_0 n)$

$$y[n] = h[n] * x[n] = \frac{\sin(\hat{\omega}_{co} n)}{\pi n} * \cos(\hat{\omega}_0 n) \quad ??$$

Multiply *the spectrum of the input times the DTFT of the filter to get*

$$y[n] = \begin{cases} \cos(\hat{\omega}_0 n) & \hat{\omega}_0 \le \hat{\omega}_{co} \\ 0 & \hat{\omega}_0 > \hat{\omega}_{co} \end{cases}$$

IDEAL HighPass Filter (HPF)



One in the passband
Zero in the stopband

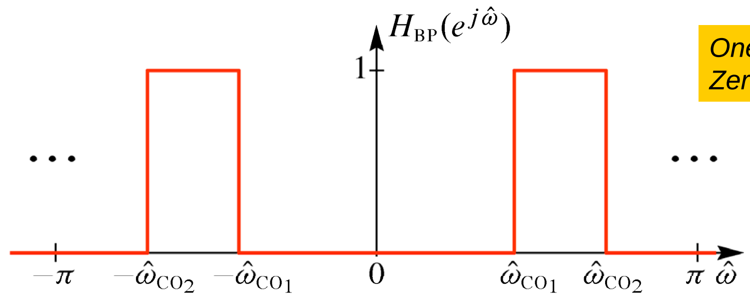
HPF is 1 minus LPF

Inverse DTFT of 1 is a delta

$$h_{HP}[n] = \delta[n] - \frac{\sin(\hat{\omega}_{co} n)}{\pi n} \quad \overset{DTFT}{\Leftrightarrow}$$

$$H_{HP}(e^{j\hat{\omega}}) = \begin{cases} 0, & |\hat{\omega}| \le \hat{\omega}_{co} \\ 1, & \hat{\omega}_{co} < |\hat{\omega}| \le \pi \end{cases}$$

IDEAL BandPass Filter (BPF)



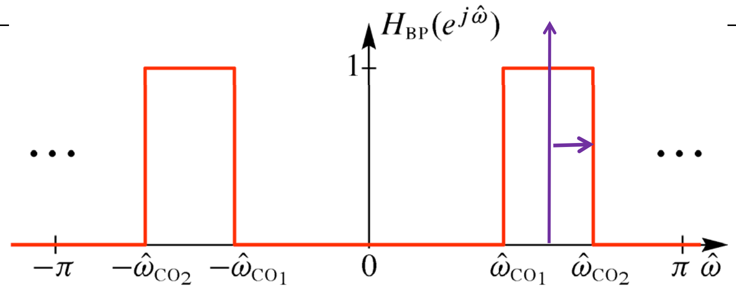
One in the passband
Zero in the 2 stopbands

BPF has two stopbands

Band Reject Filter has one stopband and two passbands.
It is one minus BPF

$$H_{BP}(e^{j\hat{\omega}}) = \begin{cases} 0 & |\hat{\omega}| \le \hat{\omega}_{co_1} \\ 1 & \hat{\omega}_{co_1} < |\hat{\omega}| \le \hat{\omega}_{co_2} \\ 0 & \hat{\omega}_{co_2} < |\hat{\omega}| \le \pi \end{cases}$$

Make IDEAL BPF from LPF



BPF is frequency shifted version of LPF

*Frequency shifting **up and down** is done by cosine multiplication in the time domain*

$$h_{\rm BP}[n] = 2\cos(\hat\omega_{\rm mid} n)\frac{\sin(\frac{1}{2}\hat\omega_{\rm diff} n)}{\pi n}$$

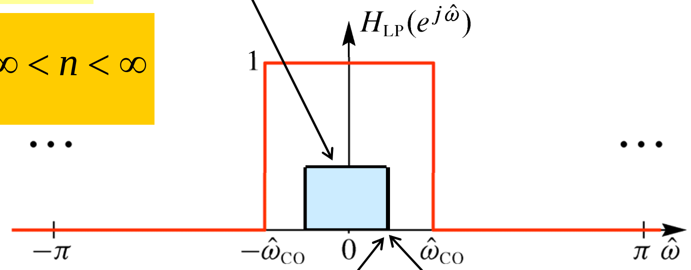
$$\overset{DTFT}{\Leftrightarrow} H_{\rm BP}(e^{j\hat\omega}) = \begin{cases} 0 & |\hat\omega| \le \hat\omega_{\rm co_1} \\ 1 & \hat\omega_{\rm co_1} < |\hat\omega| \le \hat\omega_{\rm co_2} \\ 0 & \hat\omega_{\rm co_2} < |\hat\omega| \le \pi \end{cases}$$

LPF Example 1: x[n] = sinc

$$x[n] = \frac{\sin(\hat\omega_b n)}{3\pi n}, \quad -\infty < n < \infty$$

Find output when $\hat\omega_b < \hat\omega_{\rm co}$

$$h[n] = \frac{\sin(\hat\omega_{\rm co} n)}{\pi n}, \quad -\infty < n < \infty$$



$$y[n] = \frac{\sin(\hat\omega_b n)}{3\pi n}, \quad \text{for } |\hat\omega_b| < |\hat\omega_{\rm co}|$$

$|\hat\omega_b| < |\hat\omega_{\rm co}|$

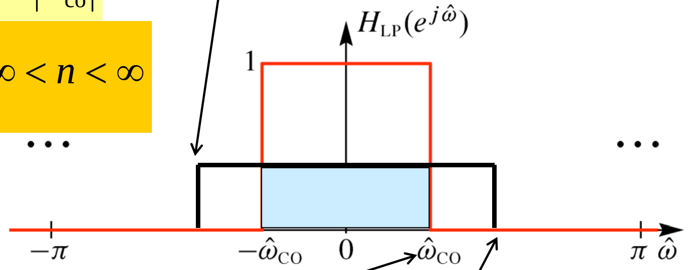
Input bandwidth Less than Filter's passband

LPF Example 2: x[n] = sinc

$$x[n] = \frac{\sin(\hat\omega_b n)}{3\pi n}, \quad -\infty < n < \infty$$

Find output when $|\omega_b| > |\omega_{\rm co}|$

$$h[n] = \frac{\sin(\hat\omega_{\rm co} n)}{\pi n}, \quad -\infty < n < \infty$$



$$y[n] = \frac{\sin(\hat\omega_{\rm co} n)}{3\pi n}, \quad \text{for } |\hat\omega_b| > |\hat\omega_{\rm co}|$$

$|\hat\omega_b| > |\hat\omega_{\rm co}|$

Filter chops off Signal bandwidth Outside of passband

Want to call the DTFT the spectrum

- The DTFT provides a ***frequency-domain*** representation
- The spectrum (Ch. 3) consists of lines at various frequencies $\hat\omega_k$, with complex amplitudes a_k
- The spectrum represents a signal $x[n]$ that is the sum of complex exponentials

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{j\hat\omega_k n}, \quad \text{where } a_k = |a_k| e^{j\angle a_k}$$

- In what sense is the DTFT going to give a sum of complex exponentials ?

The Inverse DTFT is "sum" of complex exps

- The inverse DTFT is an integral

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}\,d\hat{\omega}$$

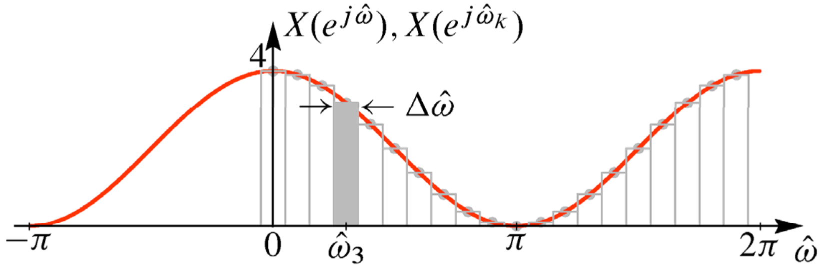
- An integral is a "sum", i.e., the limit of Riemann sums:

$$x[n] = \lim_{N\to\infty}\sum_{k=0}^{N-1}\left(X(e^{j\hat{\omega}_k})\tfrac{\Delta\hat{\omega}}{2\pi}\right)e^{j(2\pi k/N)n}$$

- The finite sum consists of cexps at frequencies $\hat{\omega}_k = (\frac{2\pi}{N})k$ with complex amplitudes $a_k = (\frac{\Delta\hat{\omega}}{2\pi})X(e^{j\omega_k})$
- The limit of these "finite spectra" is the inverse DTFT

Example: DTFT SAMPLES as a Spectrum

$$x[n] = \delta[n+1] + 2\delta[n] + \delta[n-1] \quad \Leftrightarrow \quad X(e^{j\hat{\omega}}) = 2(1+\cos\hat{\omega})$$



Samples of DTFT are Proportional to Height of Spectral Lines

$$\hat{\omega}_3 = 3(\tfrac{2\pi}{N})$$

$$a_3 = (\tfrac{\Delta\hat{\omega}}{2\pi})X(e^{j\omega_3})$$

$$x[n] = \lim_{N\to\infty}\sum_{k=0}^{N-1}\left(\tfrac{\Delta\hat{\omega}}{2\pi}X(e^{j\hat{\omega}_k})\right)e^{j(2\pi k/N)n}$$