DSP First, 2/e

Lecture 17

DFT: Discrete Fourier Transform

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, Sections 8-1, 8-2 and 8-4

LECTURE OBJECTIVES

- **D**iscrete **F**ourier **T**ransform

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}$$

- DFT from DTFT by **frequency sampling**
- DFT computation (FFT)
- DFT pairs and properties
  - Periodicity in DFT (time & frequency)

Sample the DTFT → DFT

- Want **computable** Fourier transform
  - Finite signal length (L)
  - Finite number of frequencies

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\hat{\omega}n} \Rightarrow X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n] e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = (2\pi/N)k, \quad k = 0,1,2,\ldots N-1$$

k is the frequency index

$$\text{Periodic}: X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}}) \quad \Rightarrow \quad X[k+N] = X[k]$$

Want a Computable INVERSE Fourier Transform

- Write the inverse DTFT as a finite Riemann sum:

$$x[n] = \lim_{N\to\infty} \sum_{k=0}^{N-1} \left(X(e^{j\hat{\omega}_k}) \tfrac{\Delta\hat{\omega}}{2\pi} \right) e^{j(2\pi k/N)n}$$

- Note that $\left(\frac{\Delta\hat{\omega}}{2\pi}\right) = \left(\frac{2\pi/N}{2\pi}\right) = \frac{1}{N}$
- Propose:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi k/N)n}, \quad \text{where } X[k] = X(e^{j\hat{\omega}_k})$$

- This is the inverse Discrete Fourier Transform (IDFT)

$$\Rightarrow X[k] = \sum_{k=0}^{N-1} x[n] e^{-j(2\pi k/N)n}, \quad \text{will be forward DFT}$$

Inverse DFT when L=N (proof)

- Complex exponentials are <u>ORTHOGONAL</u>

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{m=0}^{N-1} x[m] e^{-j(2\pi/N)km} \right) e^{j(2\pi/N)kn}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} x[m] \left(\sum_{k=0}^{N-1} e^{-j(2\pi/N)km} e^{j(2\pi/N)kn} \right)$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} x[m] \left(\sum_{k=0}^{N-1} e^{j(2\pi/N)k(n-m)} \right) = x[n]$$

$$= N\delta[n-m] = \begin{cases} N & n = m \\ 0 & n \neq m \end{cases}$$

Orthogonality of Complex Exponentials

The sequence set: $\left\{ e^{j(2\pi/N)kn} \right\}_{k=0}^{N-1}$ for $n = 0, 1, \cdots, N-1$

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{j(2\pi/N)kn} e^{-j(2\pi/N)mn} = \frac{1}{N} \sum_{n=0}^{N-1} e^{j(2\pi/N)n(k-m)}$$

$$= \frac{1}{N} \frac{1 - e^{j2\pi(k-m)}}{1 - e^{j(2\pi/N)(k-m)}} = \begin{cases} 1, & k = m \\ 0, & \text{otherwise} \end{cases}$$

because $0 \le k, m < N \quad |k - m| < N$

and $\lim_{l\to 0} \frac{1 - e^{j2\pi l}}{1 - e^{j(2\pi/N)l}} = N$

4-pt DFT: Numerical Example

- Take the 4-pt DFT of the following signal

$$x[n] = \delta[n] + \delta[n-1] \qquad \{x[n]\} = [1, 1, 0, 0]$$

$$X[0] = x[0]e^{-j0} + x[1]e^{-j0} + x[2]e^{-j0} + x[3]e^{-j0} = 1 + 1 + 0 + 0 = 2$$

$$X[1] = x[0]e^{-j0} + x[1]e^{-j\pi/2} + x[2]e^{-j2\pi/2} + x[3]e^{-j3\pi/2}$$
$$= 1 - j = \sqrt{2}e^{-j\pi/4}$$

$$X[2] = x[0]e^{-j0} + x[1]e^{-j\pi} + x[2]e^{-j2\pi} + x[3]e^{-j3\pi} = 1 - 1 + 0 + 0 = 0$$

$$X[3] = x[0]e^{-j0} + x[1]e^{-j3\pi/2} + x[2]e^{-j3\pi} + x[3]e^{-j9\pi/2}$$
$$= 1 + j = \sqrt{2}e^{j\pi/4}$$

N-pt DFT: Numerical Example

- Take the N-pt DFT of the impulse

$$x[n]=\delta[n] \qquad \{x[n]\}=[1,0,0,\ldots 0]$$

$$X[k]=\sum_{n=0}^{N-1}\delta[n]e^{-j(2\pi/N)kn}$$

$$=\sum_{n=0}^{0}\delta[n]e^{-j(2\pi/N)kn}=1$$

$$\{X[k]\}=[1,1,1,\ldots,1]$$

4-pt iDFT: Numerical Example

Example 66-8: Short-Length IDFT

The 4-point DFT in Example 66-7 is $X[k]=\{2,\ \sqrt{2}e^{-j\pi/4},\ 0,\ \sqrt{2}e^{j\pi/4}\}$. If we compute the 4-point IDFT of the sequence $X[k]$, we should recover $x[n]$ when we apply the IDFT summation (66.52) for each value of $n=0,1,2,3$. As before, the exponents in (66.52) will all be integer multiples of $\pi/2$ when $N=4$.

$$\begin{aligned}
x[0] &= \tfrac{1}{4}\left(X[0]e^{j0}+X[1]e^{j0}+X[2]e^{j0}+X[3]e^{j0}\right)\\
&= \tfrac{1}{4}\left(2+\sqrt{2}e^{-j\pi/4}+0+\sqrt{2}e^{j\pi/4}\right)=1\\
x[1] &= \tfrac{1}{4}\left(X[0]e^{j0}+X[1]e^{j\pi/2}+X[2]e^{j\pi}+X[3]e^{j3\pi/2}\right)\\
&= \tfrac{1}{4}\left(2+\sqrt{2}e^{j(-\pi/4+\pi/2)}+0+\sqrt{2}e^{j(\pi/4+3\pi/2)}\right)=\tfrac{1}{4}(2+(1+j)+(1-j))=1\\
x[2] &= \tfrac{1}{4}\left(X[0]e^{j0}+X[1]e^{j\pi}+X[2]e^{j2\pi}+X[3]e^{j3\pi}\right)\\
&= \tfrac{1}{4}\left(2+\sqrt{2}e^{j(-\pi/4+\pi)}+0+\sqrt{2}e^{j(\pi/4+3\pi)}\right)=\tfrac{1}{4}(2+(-1+j)+(-1-j))=0\\
x[3] &= \tfrac{1}{4}\left(X[0]e^{j0}+X[1]e^{j3\pi/2}+X[2]e^{j3\pi}+X[3]e^{j9\pi/2}\right)\\
&= \tfrac{1}{4}\left(2+\sqrt{2}e^{j(-\pi/4+3\pi/2)}+0+\sqrt{2}e^{j(\pi/4+9\pi/2)}\right)=\tfrac{1}{4}(2+(-1-j)+(-1+j))=0
\end{aligned}$$

Thus we recover the signal $x[n]=\{1,\ 1,\ 0,\ 0\}$ from its DFT coefficients, $X[k]=\{2,\ \sqrt{2}e^{-j\pi/4},\ 0,\ \sqrt{2}e^{j\pi/4}\}$.

Matrix Form for N-pt DFT

- In MATLAB, NxN DFT matrix is **`dftmtx(N)`**
  - Obtain DFT by **`X = dftmtx(N)*x`**
  - Or, **more efficiently by** **`X = fft(x,N)`**
  - Fast Fourier transform (FFT) algorithm later

$$\begin{bmatrix} X[0]\\ X[1]\\ X[2]\\ \vdots\\ X[N-1]\end{bmatrix}=\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1\\ 1 & e^{-j2\pi/N} & e^{-j4\pi/N} & \cdots & e^{-j2(N-1)\pi/N}\\ 1 & e^{-j4\pi/N} & e^{-j8\pi/N} & \cdots & e^{-j4(N-1)\pi/N}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & e^{-j2(N-1)\pi/N} & e^{-j4(N-1)\pi/N} & \cdots & e^{-j2(N-1)(N-1)\pi/N}\end{bmatrix}}_{\text{DFT matrix}}\begin{bmatrix} x[0]\\ x[1]\\ x[2]\\ \vdots\\ x[N-1]\end{bmatrix}$$

DFT matrix

Signal vector

FFT: Fast Fourier Transform

- FFT is an **algorithm** for computing the DFT
- $N\log_2 N$ versus N^2 operations
  - Count multiplications (and additions)
  - For example, when N = 1024 = 2^{10}
  - ≈10,000 ops vs. ≈1,000,000 operations
  - ≈1000 times faster
- What about N=256, how much faster?

Zero-Padding gives denser FREQUENCY SAMPLING

- Want many samples of DTFT
  - WHY? to make a smooth plot
  - Finite signal length (L)
  - Finite number of frequencies (N)
  - Thus, we need $L < N, \quad N \to \infty, \quad X[k] \to X(e^{j\hat{\omega}})$

$$X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n] e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = (2\pi / N)k, \quad k = 0,1,2,\ldots N-1$$

Zero-Padding with the FFT

- Get many samples of DTFT
  - Finite signal length (L)
  - Finite number of frequencies (N)
  - Thus, we need $L < N, \quad N \to \infty, \quad X[k] \to X(e^{j\hat{\omega}})$

In MATLAB

- Use `X = fft(x,N)`
- With `L=length(x)` less than N
- Define `xpadtoN = [x,zeros(1,N-L)];`
- Take the N-pt DFT of `xpadtoN`

DFT periodic in k (frequency domain)

- Since DTFT is periodic in frequency, the DFT must also be periodic in k

$$X[k] = X(e^{j(2\pi/N)k})$$

$$X[k+N] = X(e^{j(2\pi/N)(k+N)}) = X(e^{j(2\pi/N)(k)+(2\pi/N)N}) = X(e^{j(2\pi/N)k})$$

- What about Negative indices and Conjugate Symmetry?

$$X(e^{-j(2\pi/N)k}) = X^*(e^{j(2\pi/N)k})$$

$$\Rightarrow X[-k] = X^*[k]$$

$$X[N-k] = X^*[k]$$

$$N = 32 \quad \Rightarrow$$

$$X[31] = X^*[1]$$

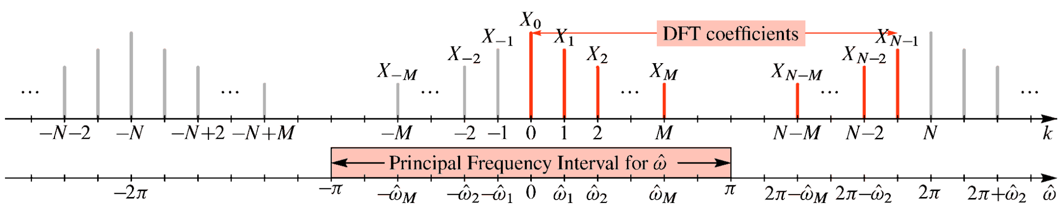
$$X[30] = X^*[2]$$

$$X[29] = X^*[3]$$

DFT Periodicity in Frequency Index

$$X[k] = X(e^{j\hat{\omega}_k}) = X(e^{j(2\pi/N)k})$$

$$k = 0,1,2,\ldots N-1$$



$$X[k+N] = X[k] \Leftrightarrow X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}})$$

$$\Rightarrow X[N-k] = X[-k],$$

e.g., $X[N-2] = X[-2]$

DFT pairs & properties

- Recall DTFT pairs because DFT is sampled DTFT
  - See next two slides
- DFT acts on a finite-length signal, so we can use DTFT pairs & properties for finite signals
- <u>Want DFT properties related to computation</u>
- And, we will concentrate on one more pair:
  - DTFT and DFT of finite sinusoid (or cexp)
    - Length-L signal
    - N-pt DFT

Table of DTFT Pairs	
Time-Domain: $x[n]$	***Frequency-Domain:*** $X(e^{j\hat{\omega}})$
$\delta[n]$	1
$\delta[n-n_d]$	$e^{-j\hat{\omega}n_d}$
$u[n]-u[n-L]$	$\dfrac{\sin(\frac{1}{2}L\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}e^{-j\hat{\omega}(L-1)/2}$
$\dfrac{\sin(\hat{\omega}_b n)}{\pi n}$	$u(\hat{\omega}+\hat{\omega}_b)-u(\hat{\omega}-\hat{\omega}_b)=\begin{cases}1 & \lvert\hat{\omega}\rvert\le\hat{\omega}_b\\ 0 & \hat{\omega}_b<\lvert\hat{\omega}\rvert\le\pi\end{cases}$
$a^n u[n]\quad(\lvert a\rvert<1)$	$\dfrac{1}{1-ae^{-j\hat{\omega}}}$
$b^n u[-n]\quad(\lvert b\rvert>1)$	$\dfrac{1}{1-be^{-j\hat{\omega}}}$

These 3 signals have infinite length

Table 8-1 Basic discrete Fourier transform pairs.

Table of DFT Pairs	
Time-Domain: $x[n]$	***Frequency-Domain:*** $X[k]$
$\delta[n]$	1
$\delta[n-n_d]$	$e^{-j(2\pi k/N)n_d}$
$r_L[n]=u[n]-u[n-L]$	$\underbrace{\dfrac{\sin(\frac{1}{2}L(2\pi k/N))}{\sin(\frac{1}{2}(2\pi k/N))}}_{=D_L(2\pi k/N)}e^{-j(2\pi k/N)(L-1)/2}$
$r_L[n]\,e^{j(2\pi k_0/N)n}$	$D_L(2\pi(k-k_0)/N)\,e^{-j(2\pi(k-k_0)/N)(L-1)/2}$

Table of DTFT Properties		
Property Name	***Time-Domain:*** $x[n]$	***Frequency-Domain:*** $X(e^{j\hat{\omega}})$
Periodic in $\hat{\omega}$		$X(e^{j(\hat{\omega}+2\pi)})=X(e^{j\hat{\omega}})$
Linearity	$ax_1[n]+bx_2[n]$	$aX_1(e^{j\hat{\omega}})+bX_2(e^{j\hat{\omega}})$
Conjugate Symmetry	$x[n]$ is real	$X(e^{-j\hat{\omega}})=X^*(e^{j\hat{\omega}})$
Conjugation	$x^*[n]$	$X^*(e^{-j\hat{\omega}})$
Time-Reversal	$x[-n]$	$X(e^{-j\hat{\omega}})$
Delay	$x[n-n_d]$	$e^{-j\hat{\omega}n_d}X(e^{j\hat{\omega}})$
Frequency Shift	$x[n]e^{j\hat{\omega}_0 n}$	$X(e^{j(\hat{\omega}-\hat{\omega}_0)})$
Modulation	$x[n]\cos(\hat{\omega}_0 n)$	$\frac{1}{2}X(e^{j(\hat{\omega}-\hat{\omega}_0)})+\frac{1}{2}X(e^{j(\hat{\omega}+\hat{\omega}_0)})$
Convolution	$x[n]*h[n]$	$X(e^{j\hat{\omega}})H(e^{j\hat{\omega}})$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty}\lvert x[n]\rvert^2$	$\frac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(e^{j\hat{\omega}})\rvert^2 d\hat{\omega}$

These 3 properties involve circular indexing

Table 8-2 Basic discrete Fourier transform properties.

Table of DFT Properties		
Property Name	*Time-Domain: $x[n]$*	*Frequency-Domain: $X[k]$*
Periodic	$x[n] = x[n+N]$	$X[k] = X[k+N]$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
Conjugate Symmetry	$x[n]$ is real	$X[N-k] = X^*[k]$
Conjugation	$x^*[n]$	$X^*[N-k]$
Time-Reversal	$x[((N-n))_N]$	$X[N-k]$
Delay	$x[((n-n_d))_N]$	$e^{-j(2\pi k/N)n_d} X[k]$
Frequency Shift	$x[n]e^{j(2\pi k_0/N)n}$	$X[k-k_0]$
Modulation	$x[n]\cos((2\pi k_0/N)n)$	$\frac{1}{2}X[k-k_0] + \frac{1}{2}X[k+k_0]$
Convolution	$\sum_{m=0}^{N-1} h[m]x[((n-m))_N]$	$H[k]X[k]$
Parseval's Theorem	$\sum_{n=0}^{N-1} \|x[n]\|^2 = \frac{1}{N}\sum_{k=0}^{N-1} \|X[k]\|^2$	

Convolution Property not the same

- Almost true for DFT:
  - Convolution maps to multiplication of transforms
- Need a different kind of convolution
  - CIRCULAR CONVOLUTION
  - LATER in an advanced DSP course
- Likewise, for Time-Shifting
  - Has to be circular
  - Because the "n" domain is also periodic

Delay Property of DFT

- Recall DTFT property for time shifting:

$$y[n] = x[n-n_d] \Leftrightarrow Y(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}})e^{-j\hat{\omega}n_d}$$

- Expected DFT property via **frequency sampling**

$$y[n] = x[\underbrace{n-n_d}] \Leftrightarrow Y[k] = X[k]e^{-j(2\pi k/N)n_d}$$

- Indices such as $n - n_d$ must be evaluated modulo-N because

$$e^{-j(2\pi k/N)(n_d+N)} = e^{-j(2\pi k/N)n_d}$$

DTFT of a Length-L Pulse

- Know DTFT of finite rectangular pulse
  - Dirichlet form and a linear phase term

$$x[n] = \begin{cases} 1 & 0 \le n < L \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow X(e^{j\hat{\omega}}) = \underbrace{\frac{\sin(\frac{1}{2}L\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}} e^{-j\hat{\omega}(L-1)/2}$$

$$D_L(\hat{\omega}) = \frac{\sin(\frac{1}{2}L\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}$$

- Use frequency-sampling to get DFT

$$X[k] = \frac{\sin(\frac{1}{2}L(2\pi k/N))}{\sin(\frac{1}{2}(2\pi k/N))} e^{-j(2\pi k/N)(L-1)/2} = D_L(2\pi k/N)\, e^{-j(\pi k/N)(L-1)}$$

DTFT of a Finite Length Complex Exponential (1)

- Know DTFT of finite rectangular pulse
  - Dirichlet form and a linear phase term

$$x[n]=\begin{cases}1 & 0\le n<L\\ 0 & \text{otherwise}\end{cases} \Leftrightarrow X(e^{j\hat\omega})=\frac{\sin(\frac{1}{2}L\hat\omega)}{\sin(\frac{1}{2}\hat\omega)}e^{-j\hat\omega(L-1)/2}$$

$$D_L(\hat\omega)=\frac{\sin(\frac{1}{2}L\hat\omega)}{\sin(\frac{1}{2}\hat\omega)}$$

- Use frequency-shift property

$$y[n]=\begin{cases}e^{j\hat\omega_0 n} & 0\le n<L\\ 0 & \text{otherwise}\end{cases} \Leftrightarrow Y(e^{j\hat\omega})=\frac{\sin(\frac{1}{2}L(\hat\omega-\hat\omega_0))}{\sin(\frac{1}{2}(\hat\omega-\hat\omega_0))}e^{-j(\hat\omega-\hat\omega_0)(L-1)/2}$$

DTFT of a Finite Length Complex Exponential (2)

- Know DTFT, so we can sample in frequency

$$y[n]=\begin{cases}e^{j\hat\omega_0 n} & 0\le n<L\\ 0 & \text{otherwise}\end{cases} \Leftrightarrow Y(e^{j\hat\omega})=\frac{\sin(\frac{1}{2}L(\hat\omega-\hat\omega_0))}{\sin(\frac{1}{2}(\hat\omega-\hat\omega_0))}e^{-j(\hat\omega-\hat\omega_0)(L-1)/2}$$

- Thus, the N-point DFT is

$$y[n]=\begin{cases}e^{j\hat\omega_0 n} & 0\le n<L\\ 0 & L\le n<N\end{cases} \Leftrightarrow$$

$$Y[k]=Y(e^{j\hat\omega}) \quad \text{at } \hat\omega=\tfrac{2\pi k}{N}$$

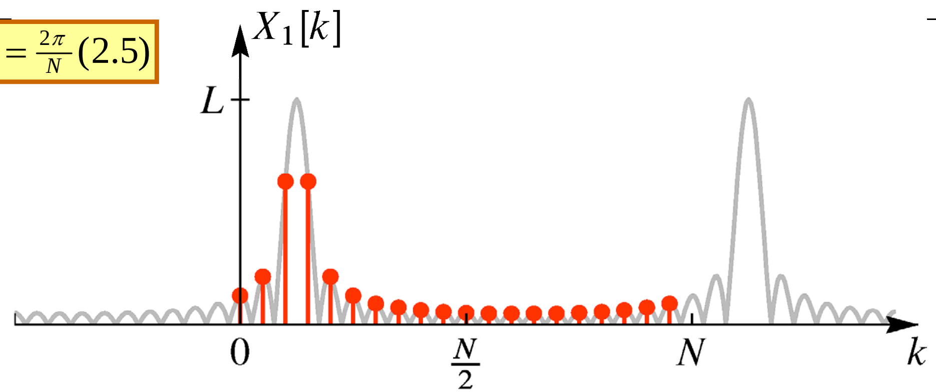
$$Y[k]=\frac{\sin(\frac{1}{2}L(\frac{2\pi k}{N}-\hat\omega_0))}{\sin(\frac{1}{2}(\frac{2\pi k}{N}-\hat\omega_0))}e^{-j(\frac{2\pi k}{N}-\hat\omega_0)(L-1)/2}$$

Dirichlet Function

$$D_L(\hat\omega)=\frac{\sin(\frac{1}{2}L\hat\omega)}{\sin(\frac{1}{2}\hat\omega)}$$

20-pt DFT of Complex Exponential

$$\hat\omega_0=\tfrac{2\pi}{N}(2.5)$$

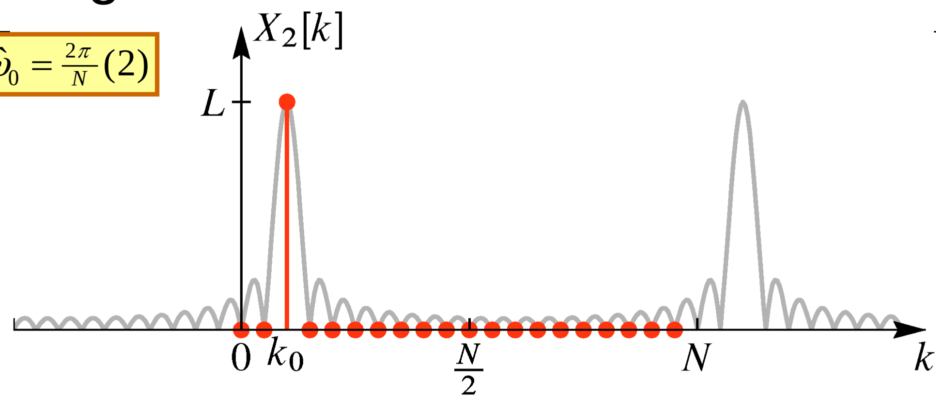


$$x_1[n]=\begin{cases}e^{j\hat\omega_0 n} & 0\le n<L\\ 0 & \text{otherwise}\end{cases} \Leftrightarrow X_1[k]=\frac{\sin(\frac{1}{2}L(\frac{2\pi k}{N}-\hat\omega_0))}{\sin(\frac{1}{2}(\frac{2\pi k}{N}-\hat\omega_0))}e^{-j(\frac{2\pi k}{N}-\hat\omega_0)(L-1)/2}$$

$$D_L(\hat\omega)=\frac{\sin(\frac{1}{2}L\hat\omega)}{\sin(\frac{1}{2}\hat\omega)}$$ so the outline is a shifted Dirichlet form

20-pt DFT of Complex Exp: “on the grid”

$$\hat\omega_0=\tfrac{2\pi}{N}(2)$$



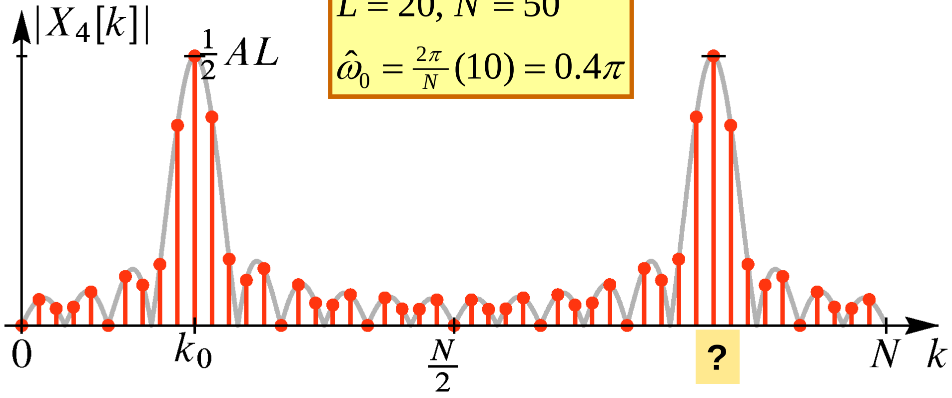
$$x_2[n]=\begin{cases}e^{j\hat\omega_0 n} & 0\le n<L\\ 0 & \text{otherwise}\end{cases} \Leftrightarrow X_2[k]=\frac{\sin(\frac{1}{2}L(\frac{2\pi k}{N}-\hat\omega_0))}{\sin(\frac{1}{2}(\frac{2\pi k}{N}-\hat\omega_0))}e^{-j(\frac{2\pi k}{N}-\hat\omega_0)(L-1)/2}$$

$$D_L(\hat\omega)=\frac{\sin(\frac{1}{2}L\hat\omega)}{\sin(\frac{1}{2}\hat\omega)}$$ so the outline is a shifted Dirichlet form

50-pt DFT of Sinusoid: zero padding

$L = 20,\ N = 50$

$\hat{\omega}_0 = \frac{2\pi}{N}(10) = 0.4\pi$

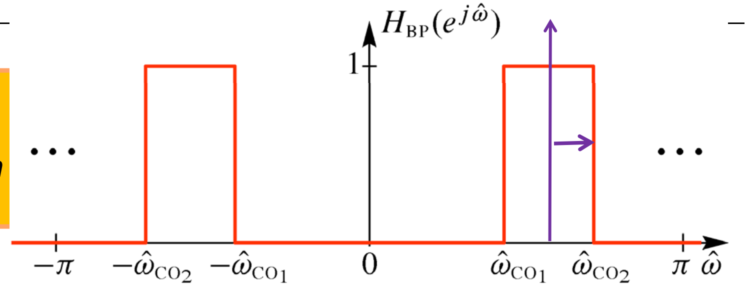


$$x_4[n] = \begin{cases} A\cos(\hat{\omega}_0 n) & 0 \le n < L \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow$$

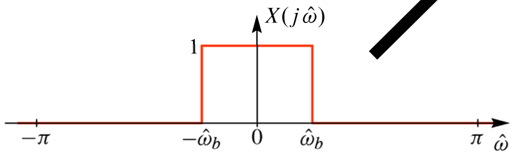
$$X_4[k] = \tfrac{1}{2} A D_L(\tfrac{2\pi k}{N} - \hat{\omega}_0)\, e^{-j(\frac{2\pi k}{N} - \hat{\omega}_0)(L-1)/2} + \tfrac{1}{2} A D_L(\tfrac{2\pi k}{N} + \hat{\omega}_0)\, e^{-j(\frac{2\pi k}{N} + \hat{\omega}_0)(L-1)/2}$$

RECALL: BandPass Filter (BPF)

Frequency shifting up and down *is done by cosine multiplication in the time domain*



BPF is frequency shifted version of LPF (below)



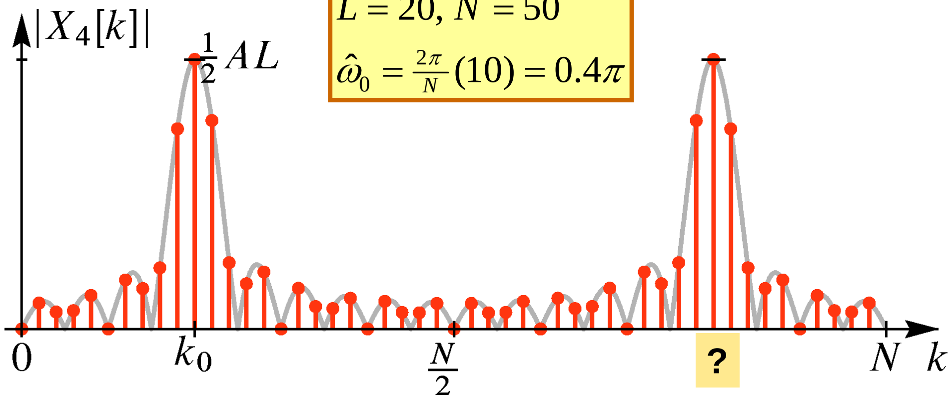
$$h_{BP}[n] = 2\cos(\hat{\omega}_{mid} n)\frac{\sin(\frac{1}{2}\hat{\omega}_{diff} n)}{\pi n}$$

$$\overset{DTFT}{\Leftrightarrow} H_{BP}(e^{j\hat{\omega}}) = \begin{cases} 0 & |\hat{\omega}| \le \hat{\omega}_{co_1} \\ 1 & \hat{\omega}_{co_1} < |\hat{\omega}| \le \hat{\omega}_{co_2} \\ 0 & \hat{\omega}_{co_2} < |\hat{\omega}| \le \pi \end{cases}$$

50-pt DFT of Sinusoid: zero padding

$L = 20,\ N = 50$

$\hat{\omega}_0 = \frac{2\pi}{N}(10) = 0.4\pi$



Zero-crossings of Dirichlet ?

Width of Dirichlet ?

Density of frequency samples?

Thus we have a simple BPF