DSP First, 2/e

Lecture 18

DFS: Discrete Fourier Series, and Windowing

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, Sections 8-3, 8-5 & 8-6

LECTURE OBJECTIVES

- **D**iscrete **F**ourier **S**eries for periodic x[n]

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn} \qquad x[n] = \sum_{k=0}^{N-1} a_k e^{j(2\pi/N)kn}$$

  - DFT of one period with scaling by 1/N gives scaled DFS coefficients
- Windowing
  - extract short sections from long signal

Review

- **D**iscrete **F**ourier **T**ransform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn} \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j(2\pi/N)kn}$$

- DFT is **frequency sampled** DTFT
  - For finite-length signals
- DFT computation via FFT
  - FFT of zero-padded signal→more freq samples
- Transform pairs & properties (DTFT & DFT)

Comparison: DFT and DTFT

Discrete Fourier Transform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

Inverse DFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}$$

Discrete-time Fourier Transform (DTFT)

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\hat{\omega}n}$$

Inverse DTFT

$$x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} X(e^{j\hat{\omega}}) e^{j\hat{\omega}n} d\hat{\omega}$$

Inverse DFT always makes a periodic signal

- Evaluate N-pt IDFT outside of [0,N-1]

$$x[n+N] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)k(n+N)}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn} e^{j(2\pi/N)k(N)}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn} = x[n]$$

- <u>Thus the IDFT synthesizes a periodic signal</u>

Fourier Series for Discrete-Time Signal

Given a periodic sequence x[n], how do we write it as a <u>sum of sinusoids</u> (or complex exponentials) ?

Which frequencies? How many? Fundamental ?

Exponentials must have the <u>same period</u> as x[n], which is N. There are only N possible exps.

$$\left\{e^{j(2\pi k/N)n}\right\}_{k=0}^{N-1} \quad \text{for } n = 0, 1, \cdots, N-1$$

$$e^{j(0)n}, e^{j(2\pi/N)n}, e^{j(4\pi/N)n}, e^{j(6\pi/N)n}, \ldots e^{j(2\pi(N)/N)n}$$

Discrete Fourier Series Representation (2)

$$\left\{e^{j(2\pi k/N)n}\right\}_{k=0}^{N-1} \quad \text{for } n = 0, 1, \cdots, N-1$$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{j(2\pi k/N)n}$$

Given the sequence x[n], how do we find a_k?

Recall IDFT always synthesizes a periodic x[n]

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j(2\pi k/N)n}$$

$$\Rightarrow a_k = \tfrac{1}{N} X[k]$$

So, we find a_k by taking the N-pt DFT of one period of x[n] and then multiplying by 1/N

Discrete Fourier Series (DFS)

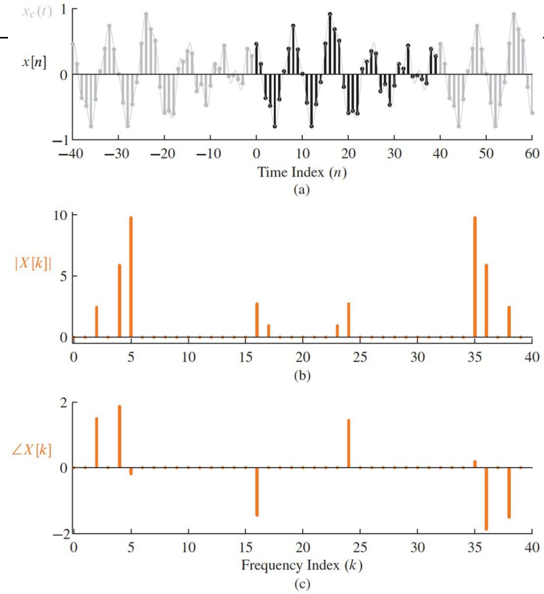
$$x[n] = \sum_{k=0}^{N-1} a_k e^{j(2\pi k/N)n}$$

Synthesis of a periodic signal x[n] = x[n+N]

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j(2\pi k/N)n}$$

Find a_k by taking N-pt DFT of one period of x[n] and then multiplying by 1/N

DFT of one period → DFS



Can you get the DFS coeffs from this DFT ?

Figure 8-12 Periodic sequence $x[n]$ in (8.43) and the corresponding 40-point DFT spectrum $X[k]$. (a) Dark region indicates the 40-point interval (starting at $n = 0$) taken for analysis. (b) DFT magnitude spectrum. (c) DFT phase spectrum.

DFS Synthesis Example

$$x[n] = \sum_{k=0}^{N-1} a_k e^{j(2\pi k/N)n}$$

$$\{a_k\} = \{1,1,1,0,0,0,1,1\} \quad \text{for } n = 0,1,2,3,4,5,6,7$$

$$x[n] = e^{j(0)n} + e^{j(2\pi/8)n} + e^{j(4\pi/8)n} + e^{j(12\pi/8)n} + e^{j(14\pi/8)n}$$

Recall negative frequencies in high indices for the DFT

$$e^{j(14\pi/8)n} = e^{j((16\pi-2\pi)/8)n} = e^{-j(2\pi/8)n}$$

$$x[n] = e^{j(0)n} + e^{j(2\pi/8)n} + e^{j(4\pi/8)n} + e^{-j(-4\pi/8)n} + e^{-j(2\pi/8)n}$$

$$= 1 + 2\cos(\tfrac{\pi}{4}n) + 2\cos(\tfrac{\pi}{2}n)$$

How is DFS related to cont-time Fourier Series ?

- Fourier Series (from Chap. 3)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi k f_0)t}, \quad \text{where } a_k = |a_k| e^{j\angle a_k}$$

- Need to obey the Nyquist rate: i.e., band-limited signal

$$x(t) = \sum_{k=-M}^{M} a_k e^{j(2\pi k f_0)t}, \quad \text{and } f_s > 2Mf_0$$

- Then sample

$$x[n] = \sum_{k=-M}^{M} a_k e^{j(2\pi k f_0)(n/f_s)}, \quad \text{and } f_s > 2Mf_0$$

$$= \sum_{k=-M}^{M} a_k e^{j(2\pi k/N)(n)}, \quad \text{if } f_s = Nf_0$$

How is DFS related to cont-time Fourier Series (2)

- Band-limited Fourier Series (from Chap. 3)

$$x(t) = \sum_{k=-M}^{M} a_k e^{j(2\pi k f_0)t}, \quad \text{and } f_s > 2Mf_0$$

- Can be sampled to give periodic x[n]

$$x[n] = \sum_{k=-M}^{M} a_k e^{j(2\pi k/N)(n)}, \quad \text{if } f_s = Nf_0$$

- Compare to DFS

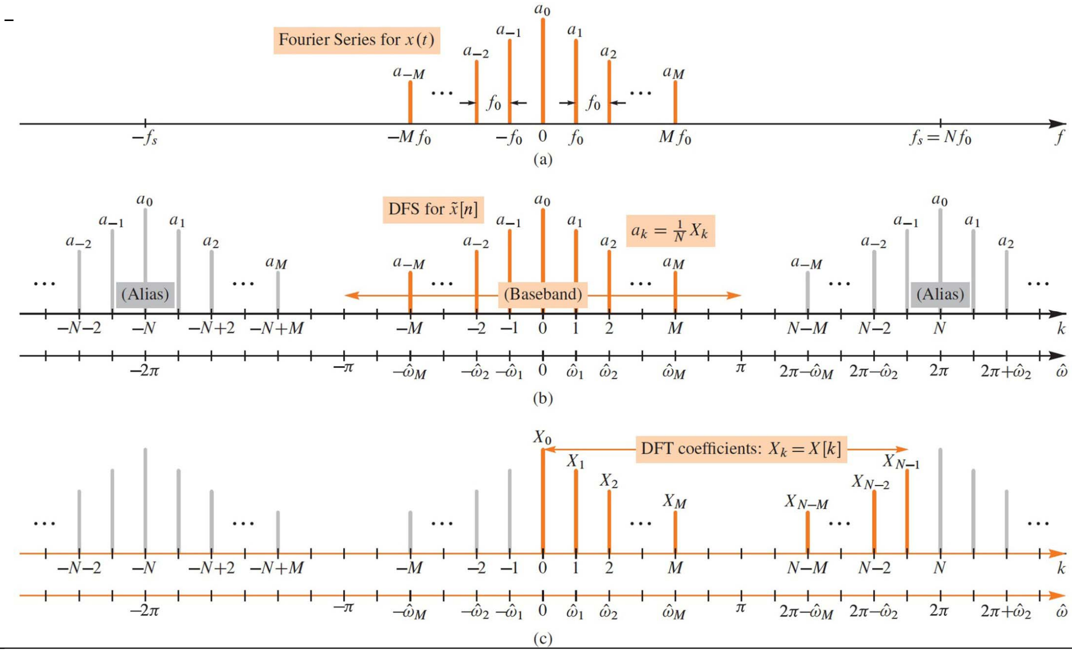
$$x[n] = \sum_{k=0}^{N-1} a_k e^{j(2\pi k/N)n}$$

- Same coefficients a_k

Recall

$\hat{\omega} f_s = \omega$

Spectrum Analysis of a Periodic Signal x[n]

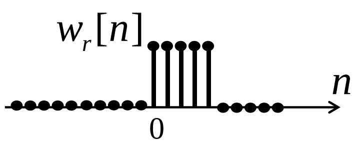


Windows

- Finite-Length signal (L) with positive values
  - Extractor
  - Truncator

Rectangular Window

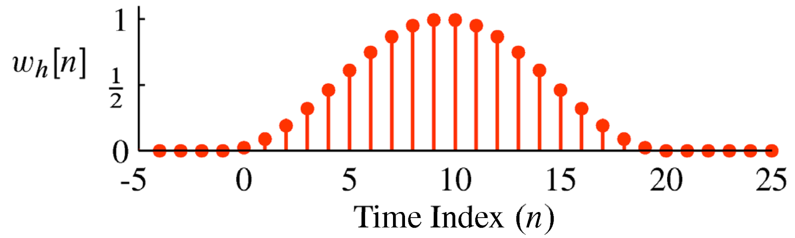
$$w_r[n] = \begin{cases} 0 & n < 0 \\ 1 & 0 \le n < L \\ 0 & n \ge L \end{cases}$$



$$w_r[n]x[n+n_0] = \begin{cases} 0 & n < 0 \\ x[n+n_0] & 0 \le n < L \\ 0 & n \ge L \end{cases}$$

Von Hann Window (Time Domain)

- Plot of Length-20 von Hann window

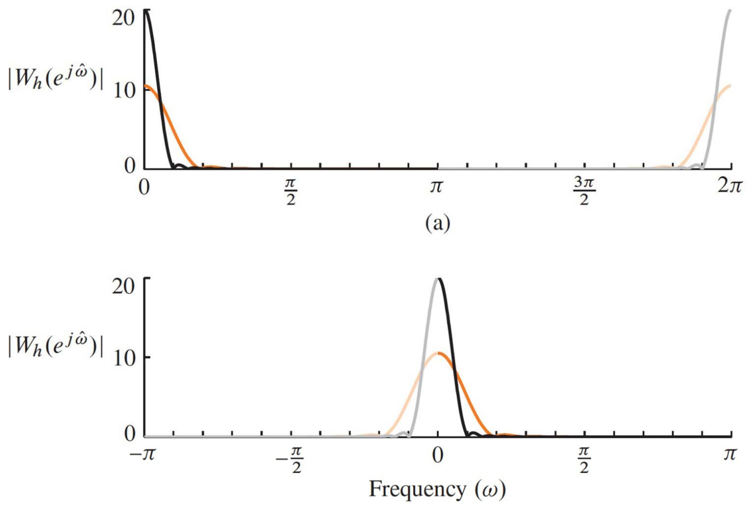


von Hann Window (Length L)

$$w_h[n] = \begin{cases} 0 & n < 0 \\ \frac{1}{2} - \frac{1}{2}\cos(2\pi(n+1)/(L+1)) & 0 \le n < L \\ 0 & n \ge L \end{cases}$$

Von Hann Window (Frequency Domain)

- DTFT (magnitude) of Length-20 Hann window



(a)

(b)

Window section of sinusoid, then DFT

- Multiply the very long sinusoid by a window

$$x[n] = A\cos(\hat{\omega}_0 n + \varphi) \quad -\infty < n < \infty$$

$$\hat{\omega}_0 = 0.4\pi$$

  - Take the N-pt DFT
    - Finite number of frequencies (N)
  - Finite signal length (L) = window length

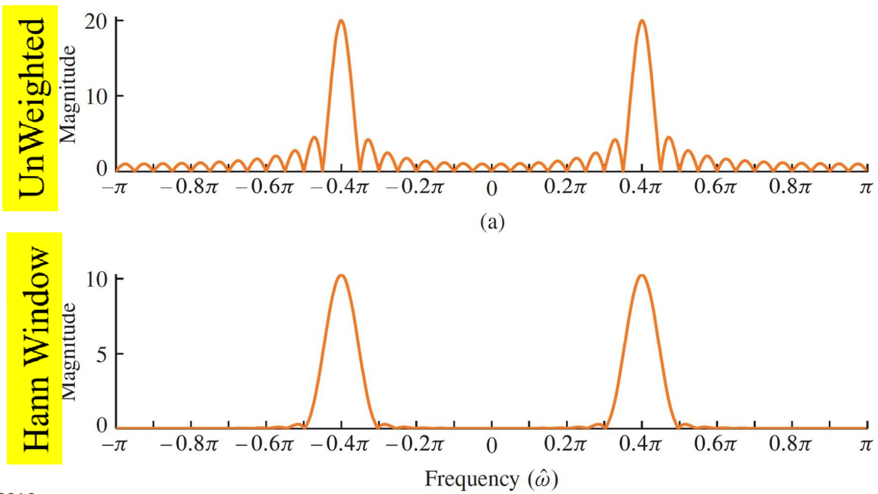
$$X_L(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} w_L[n]x[n]e^{-j\hat{\omega}n} \rightarrow \sum_{n=0}^{L-1} w_L[n]x[n]e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = (2\pi / N)k, \quad k = 0,1,2,\ldots N-1$$

Expectation: 2 narrow spectrum lines

DTFT of Windowed Sinusoid (with different windows)

- DTFT (magnitude) of windowed sinusoid
  - Length-40 Hann window vs Length-40 Rectangular window

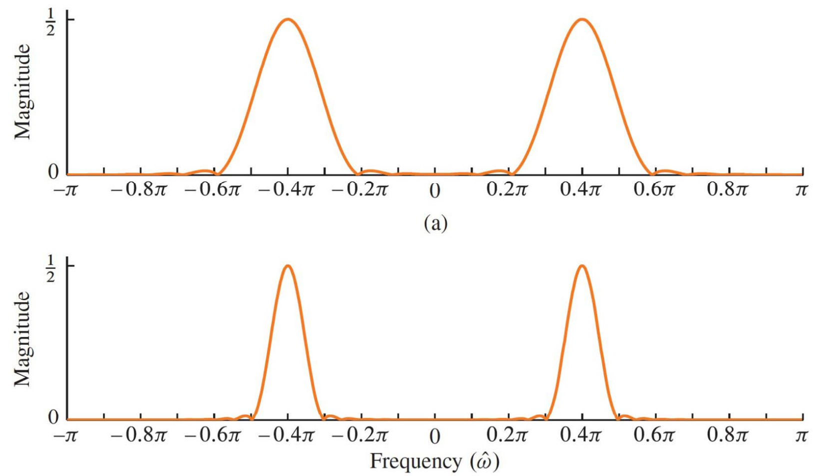


(a)

(b)

Change Window Length

- DTFT (magnitude) of windowed sinusoid.
  - Length-20 Hann window vs. Length-40 Hann window



(a)

(b)