DSP First, 2/e

Lecture 19

Spectrogram: Spectral Analysis via DFT & DTFT

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, Sects. 8-6 & 8-7
- Other Reading:
  - FFT: Chapter 8, Sect. 8-8

LECTURE OBJECTIVES

- **Spectrogram**
- Time-Dependent Fourier Transform
  - DFTs of short windowed sections
- Frequency Resolution
  - To resolve closely spaced spectrum lines
  - Need long windows

Review & Recall

- **D**iscrete **F**ourier **T**ransform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}$$

- DFT is **frequency sampled** DTFT
  - For finite-length signals
- DFT computation is actually done via FFT
  - FFT of zero-padded signal→more freq samples
- Transform pairs & properties (DTFT & DFT)

Recall: DFT and DTFT

Discrete Fourier Transform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn}$$

Inverse DFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j(2\pi/N)kn}$$

Discrete-time Fourier Transform (DTFT)

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

Inverse DTFT

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}\,d\hat{\omega}$$

The Transform Method (DTFT)

- To get the output signal y[n], given the input x[n], and the system defined by h[n].

$$y[n] = h[n] * x[n]$$

$$x[n] \rightarrow X(e^{j\hat{\omega}})$$

$$Y(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}})X(e^{j\hat{\omega}})$$

$$Y(e^{j\hat{\omega}}) \rightarrow y[n]$$

- <u>Convolution becomes multiplication of DTFTs</u>

Frequency Response Method

- To get the output signal y[n], given the input x[n] is a sinusoid, and the system defined by h[n].

$$y[n] = h[n] * A\cos(\hat{\omega}_0 n + \varphi)$$

$$h[n] \rightarrow H(e^{j\hat{\omega}})$$

$$\text{eval } H(e^{j\hat{\omega}}) \rightarrow H(e^{j\hat{\omega}_0}) = H_0$$

$$y[n] = A\cdot|H_0|\cos(\hat{\omega}_0 n + \varphi + \angle H_0)$$

- <u>Multiply Magnitudes, Add Phases</u>

DTFT, DFT and DFS are Tools for <u>Analysis</u>

- **DTFT** applies to discrete time-sequences, regardless of length (as long as x[n] is absolutely summable);
  - Note: sinusoids are not absolutely summable
- **N-pt DFT** of a finite-length sequence is equivalent to **sampling the corresponding DTFT** on frequency axis at N equally spaced points, **without losing any information**
  - **Zero-padding** when N>L
- **If the <u>sequence</u> is periodic, and the DFT is performed on one period,** the result is a discrete Fourier Series **(DFS)** which is an exact representation.

Various Situations in Signal Analysis

A. Signal property does not fluctuate with time

1. Sampled signal is periodic with **known** period N_0 – use N_0-pt DFT and get the precise result (DFS)
2. Sampled signal is aperiodic, or period is **unknown**, or hard to ascertain precisely. (COMMON CASE →)
3. Signal characteristic is global (for the entire signal)
   a. Finite-Length, e.g., pulse
   b. Infinite-Length: e.g., right-sided exponential

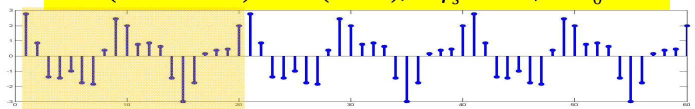
B. Sampled signal property changes with time

1. Signal defined by math formulas over intervals
2. **Recorded** signal, most likely with changing sinusoidal content, or properties – the most general case (COMMON CASE →)

Examples

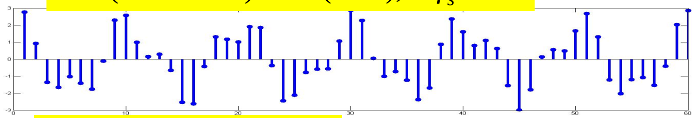
A.1 Periodic Sequence

$2\cos(20\pi t + 0.5) + \cos(50\pi t), \quad f_s = 100, \quad N_0 = 20$



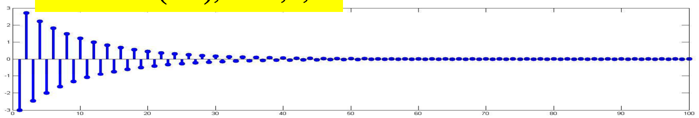
A.2 Nonperiodic Sequence

$2\cos(20\pi t + 0.5) + \cos(150 t), \quad f_s = 100$

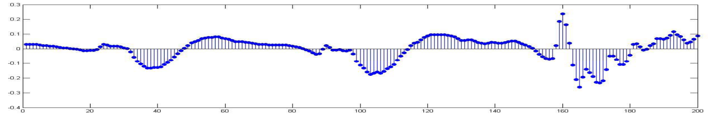


A.3 Global infinite-length sequence

$3e^{-0.1n}\cos(n\pi), n = 1,2,....$



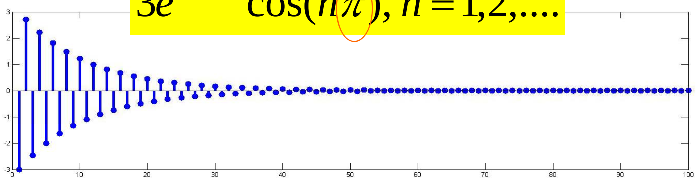
B.2 A general sequence



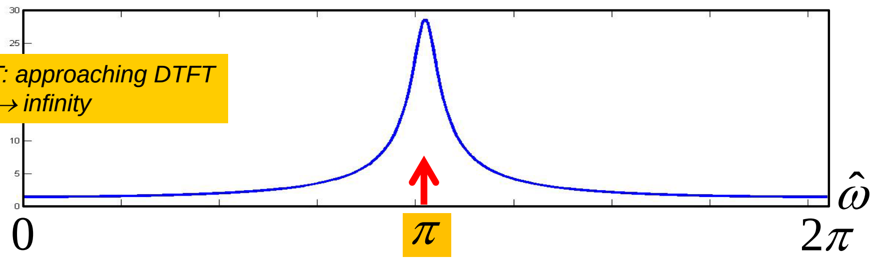
A.3: DTFT gives global spectrum

A.3 sequence from formula

$3e^{-0.1n}\cos(n\pi), n = 1,2,....$



N-pt DFT: approaching DTFT when N → infinity

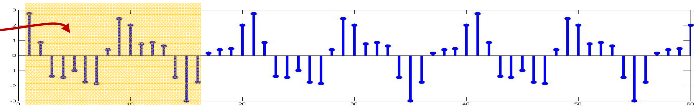


More about this later.

When Periodicity Is Not Precise – A.2

- Periodicity is imprecise – either not known exactly or impractical to take exactly one period of data; for example:

$L = 16$

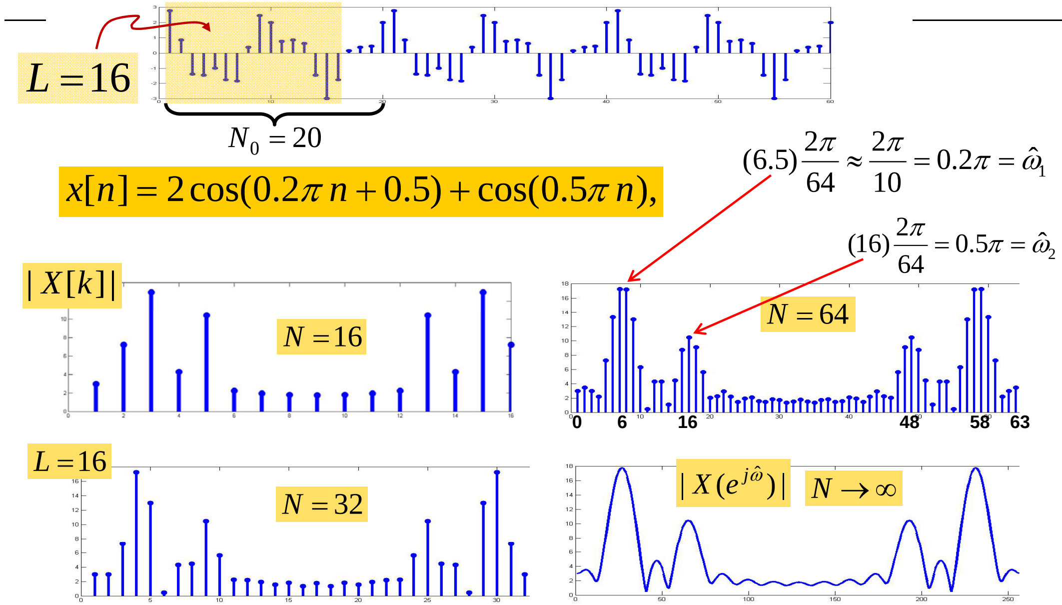


$N_0 = 20$ *Actual period of sequence*

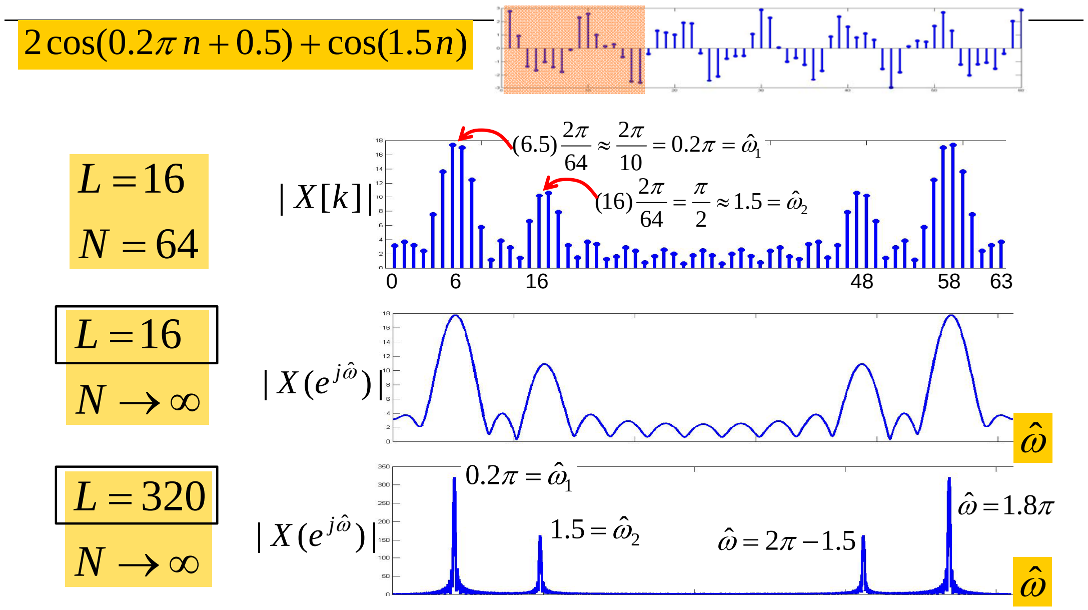
$$x(t) = 2\cos(20\pi t + 0.5) + \cos(50\pi t),$$
$$f_0 = 5, \quad f_s = 100, \quad N_0 = 20$$

- We take L data points for analysis; actual period is not L ($\neq 20$ for the example).

Imprecise Period Example (2)



Similar Example – Changing L



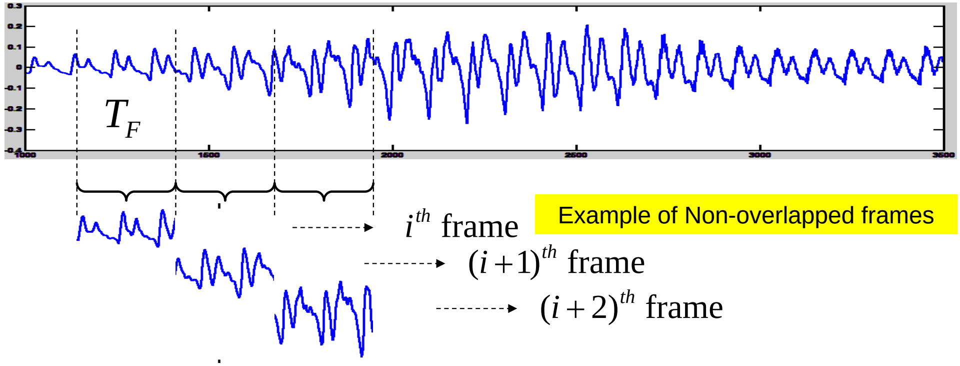
Observations

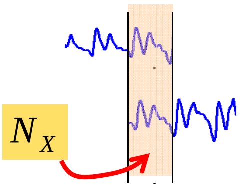
- When signal property stays fixed, the longer the section of data (larger L), the sharper the frequency resolution.
- The larger the number of points for the DFT, N, the denser the frequency axis sampling.
- However, the <u>resolution</u> (or separation) of sinusoids w.r.t. frequency <u>is determined by the data length L</u>, not by length of the N-pt DFT (with zero padding).
- When signal properties change with time, there are other considerations that limit the size of L; see case **<u>B</u>**.

<u>Case B</u> – The General Case

- DFT can also be used as a tool for finding the spectral composition of an arbitrary signal:
  - In general, we observe changing signal properties
  - **<u>Short-time</u>** analysis framework
    - Analyze sequence one block (frame) of data at a time using DFT
    - Successive blocks of data may overlap
    - "**SPECTROGRAM**"

FRAME = WINDOW of DATA





Data also can be extracted from <u>overlapped</u> frames – overlapped characteristics provide smoother transitions from one frame to next

Denote Number of Overlapping Points as N_X

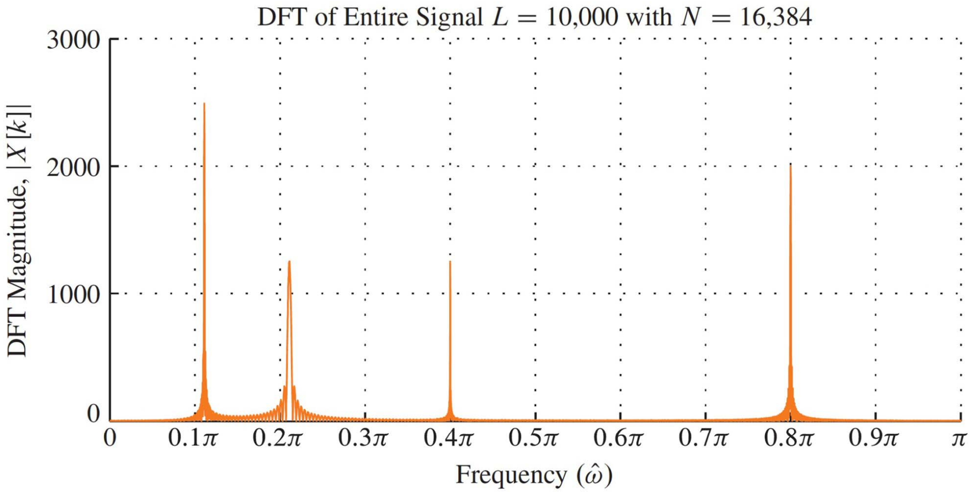
A very long signal

- Four intervals
- Four spectrum lines

$$x[n] = x(n/f_s) = \begin{cases} 5\cos(0.211\pi n) & 0 \le n \le 499 \\ 2\cos(0.111\pi n) & 500 \le n \le 2999 \\ 2\cos(0.8\pi n) & 3000 \le n \le 4999 \\ \frac{1}{2}\cos(0.4\pi n) & 5000 \le n \le 9999 \end{cases}$$

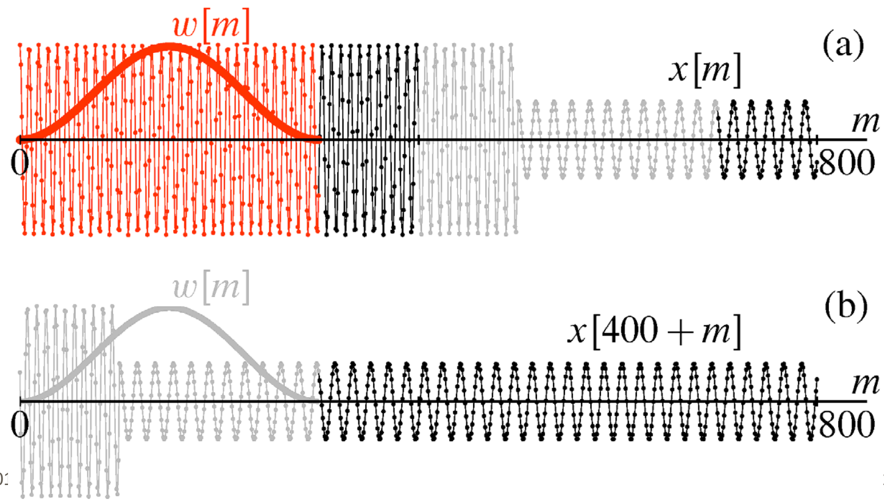
DFT of a very Long Signal (Global)

- Length 10,000 signal, 16384-pt DFT



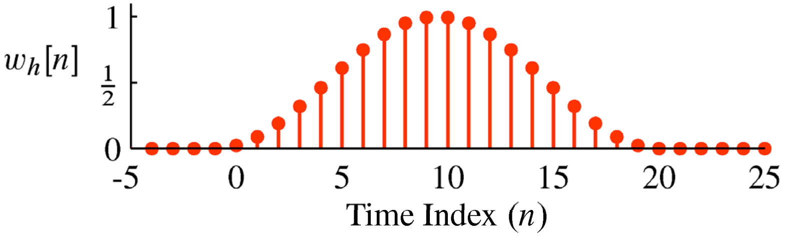
Windowing a Long Signal

- Long signal is time-shifted into the domain of the window, [0,300] in this case



Von Hann Window in Time Domain

- Plot of Length-20 von Hann window

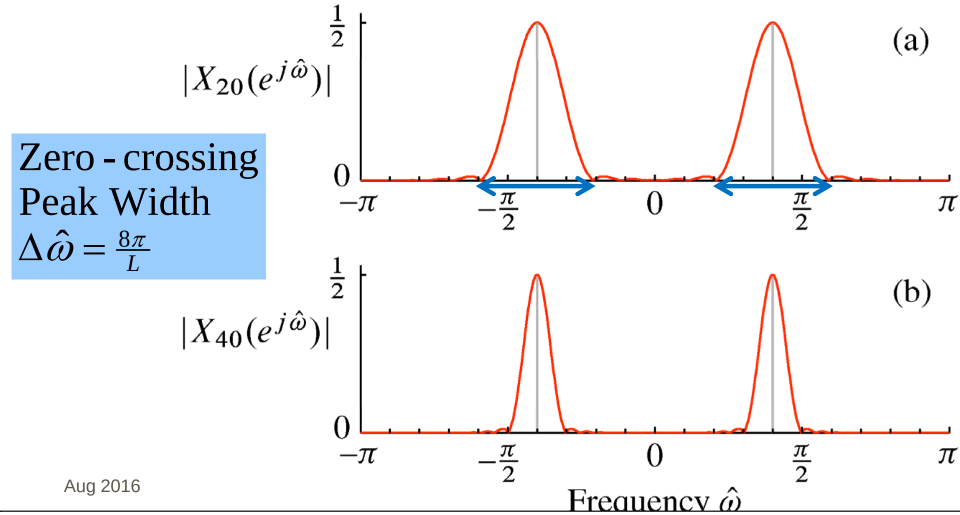


von Hann Window (Length L)

$$w_h[n] = \begin{cases} 0 & n < 0 \\ \frac{1}{2} - \frac{1}{2}\cos(2\pi(n+1)/(L+1)) & 0 \le n < L \\ 0 & n \ge L \end{cases}$$

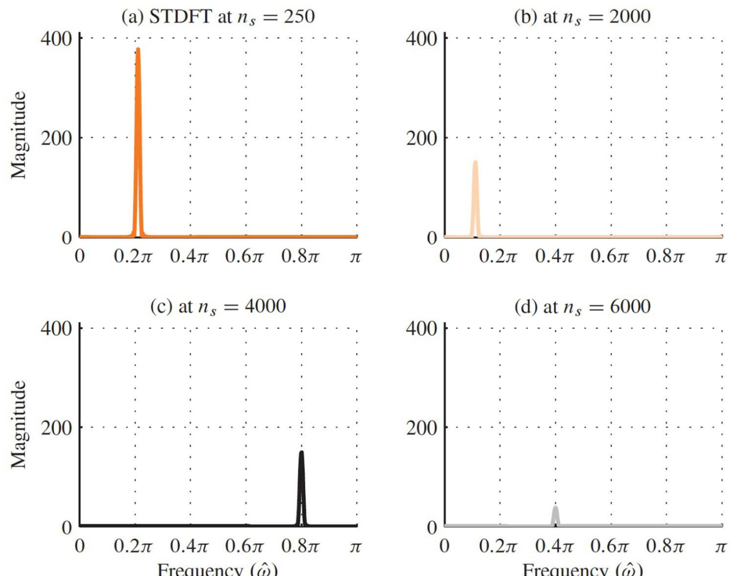
DTFT of Hann Window: Change Length

- DTFT (magnitude) of windowed sinusoid.
  - Length-20 Hann window vs. Length-40 Hann window



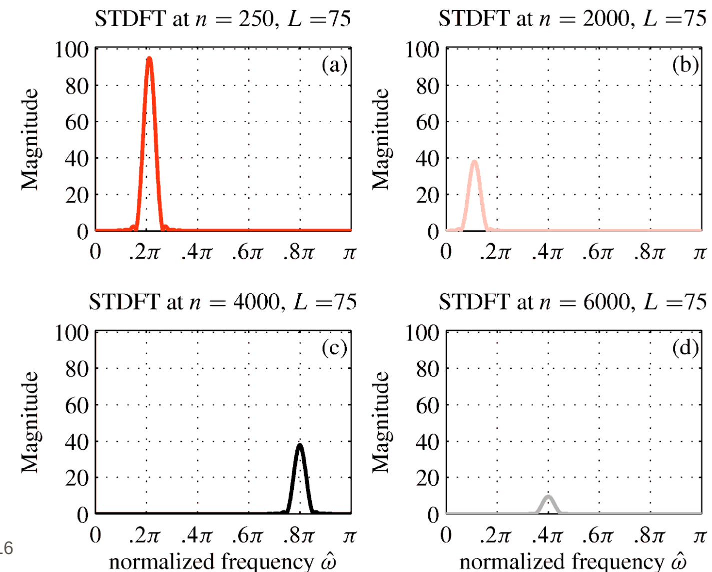
Short DFTs of Windowed Sections (L=301)

- 301-pt Hann window. 1024-pt DFT (zero-padded)



Short DFTs of Windowed Sections (L=75)

- 75-pt Hann window. 1024-pt DFT (zero-padded)



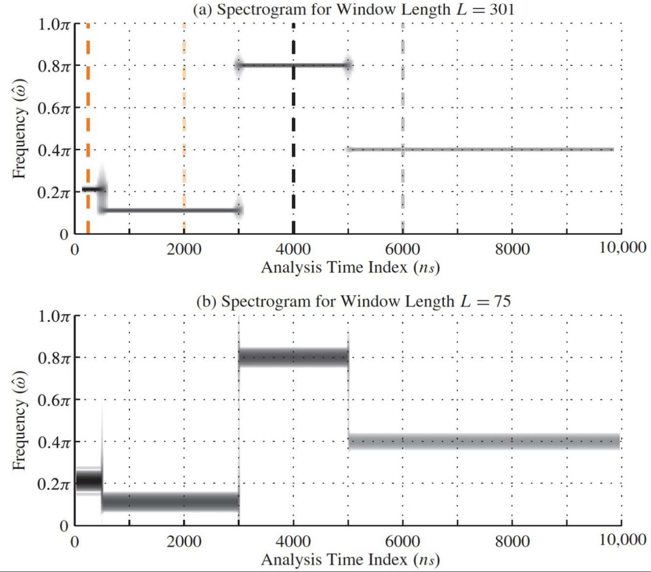
A very long signal

- Four intervals
- Four spectrum lines

$$x[n] = x(n/f_s) = \begin{cases} 5\cos(0.211\pi n) & 0 \le n \le 499 \\ 2\cos(0.111\pi n) & 500 \le n \le 2999 \\ 2\cos(0.8\pi n) & 3000 \le n \le 4999 \\ \frac{1}{2}\cos(0.4\pi n) & 5000 \le n \le 9999 \end{cases}$$

Two Spectrograms of very Long Signal

- Window Lengths of L=301 and L=75; overlap 90%



MATLAB Spectrogram

- **In MATLAB,** use

`S = spectrogram(X,WINDOW,NOVERLAP,NFFT,Fs,'yaxis')`

f_s **Fs** is the sampling frequency specified in samples/s

- **WINDOW**:

L
  - if a vector, extract segments of **X** of equal length, and then multiply window vector and data vector, element by element;
  - If an integer, same as above but use a Hamming window of the specified length
  - if unspecified, use default

N_X **NOVERLAP: (<WINDOW)**
  - Number of overlapping data between successive frames

N **NFFT:** Length of FFT

Resolution Test

- Closely spaced frequency components
  - How close can the lines be?
  - Depends on window (section) length
  - Inverse relationship

$$x[n] = x(n/f_s) = \begin{cases} \cos(0.2\pi n) + 1.5\cos(0.227\pi n) & 0 \le n \le 4999 \\ \cos(0.2\pi n) & 5000 \le n \le 6999 \\ 3\cos(0.6\pi n) + \cos(0.7\pi n) & 7000 \le n \le 9999 \end{cases}$$

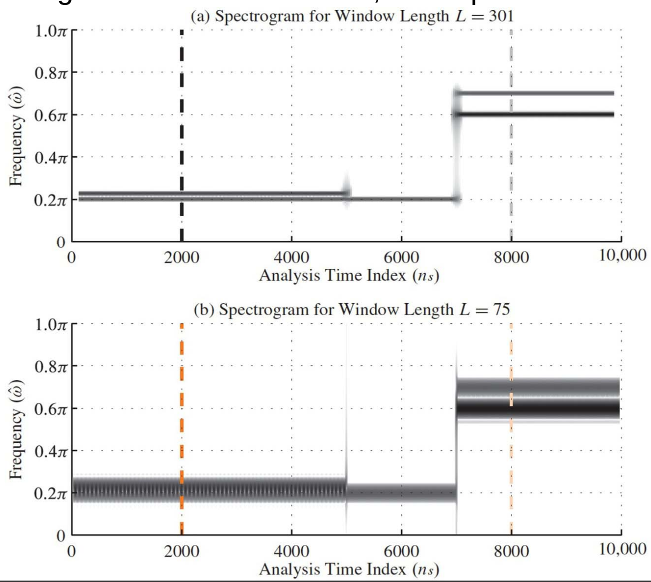
$$\cos(0.2\pi n) + 1.5\cos(0.227\pi n) \Rightarrow \Delta\hat{\omega} = 0.027\pi = 8\pi/?$$

$$3\cos(0.6\pi n) + \cos(0.7\pi n) \Rightarrow \Delta\hat{\omega} = 0.1\pi = 8\pi/80$$

$$\left(\frac{\Delta\hat{\omega}}{8\pi}\right) = \frac{1}{296}$$

Two Spectrograms: Resolution Test

- Window Lengths of L=301 and L=75, overlap 90%



Short DFTs of Windowed Sections

- 75-pt and 301-pt Hann windows. 1024-pt DFT (zero-padded)