

DSP First, 2/e

Lecture 20 Z Transforms: Introduction

READING ASSIGNMENTS

- This Lecture:
 - Chapter 9, Sects 9-1 through 9-5
- Other Reading:
 - Recitation: CASCADING SYSTEMS

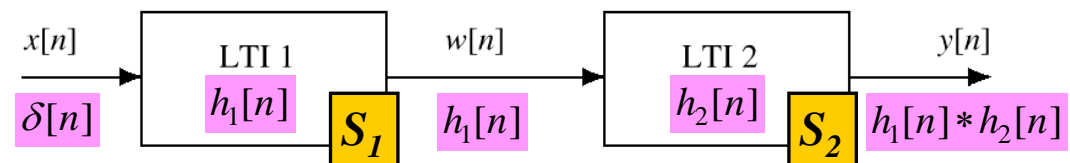
LECTURE OBJECTIVES

- INTRODUCE the Z-TRANSFORM
 - Give Mathematical Definition
 - Show how the **H(z) POLYNOMIAL** simplifies analysis
 - **CONVOLUTION** is SIMPLIFIED !
- Z-Transform can be applied to
 - FIR Filter: $h[n] \rightarrow H(z)$
 - Signals: $x[n] \rightarrow X(z)$

$$H(z) = \sum_n h[n]z^{-n}$$

MOTIVATION: CASCADE SYSTEMS

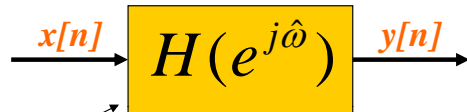
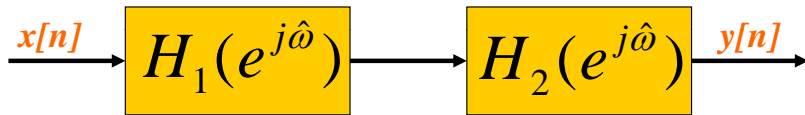
- Remember: $h_1[n] * h_2[n]$



- Would rather do **MULTIPLICATION**
- Can rearrange the order of S_1 and S_2
 - **Convolution is Commutative**

CASCADE EQUIVALENT

- MULTIPLY** the Frequency Responses



EQUIVALENT SYSTEM

$$H(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}})$$

A TALE OF TWO DOMAINS

- Time domain:**
 - Can use with ANY signal
 - Difficult to work with (e.g., cascade=convolve)
- Frequency domain:**
 - Easy to work with (e.g., cascade=multiply)
 - Can only use with sinusoids

TIME-DOMAIN

$$y[n] = \sum_{k=0}^M b_k x[n-k]$$

FREQ-DOMAIN

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^M b_k e^{-j\hat{\omega}k}$$



TWO (no, THREE) DOMAINS

Z-TRANSFORM-DOMAIN

Algebra Domain:
Best of both worlds?

POLYNOMIALS: $H(z)$

$\{b_k\}$

TIME-DOMAIN

$$y[n] = \sum_{k=0}^M b_k x[n-k]$$

FREQ-DOMAIN

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^M b_k e^{-j\hat{\omega}k}$$

TRANSFORM CONCEPT

- Move to a new domain where
 - OPERATIONS are EASIER & FAMILIAR
 - Use **POLYNOMIALS**
- TRANSFORM both ways
 - $x[n] \rightarrow X(z)$ (into the z domain)
 - $X(z) \rightarrow x[n]$ (back to the time domain)

Z-Transform DEFINITION

- POLYNOMIAL Representation of LTI SYSTEM:

$$H(z) = \sum_n h[n]z^{-n}$$

APPLIES to any SIGNAL

- EXAMPLE:

$$\{h[n]\} = \{2, 0, -3, 0, 2\}$$

$$\begin{aligned} H(z) &= 2z^{-0} + 0z^{-1} - 3z^{-2} + 0z^{-3} + 2z^{-4} \\ &= 2 - 3z^{-2} + 2z^{-4} \\ &= 2 - 3(z^{-1})^2 + 2(z^{-1})^4 \end{aligned}$$

POLYNOMIAL in z^{-1}

Z-Transform EXAMPLE

- ANY SIGNAL has a z-Transform:

$$X(z) = \sum_n x[n]z^{-n}$$

Example 7.1

n	$n < -1$	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	2	4	6	4	2	0	0

$$X(z) = ? \quad X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4}$$

Example 9.2

$$X(z) = 1 - 2z^{-1} + 3z^{-3} - z^{-5}$$

EXPONENT GIVES TIME LOCATION

$$x[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ -2 & n = 1 \\ 0 & n = 2 \\ 3 & n = 3 \\ 0 & n = 4 \\ -1 & n = 5 \\ 0 & n > 5 \end{cases}$$

$$x[n] = ?$$

$$x[n] = \delta[n] - 2\delta[n-1] + 3\delta[n-3] - \delta[n-5]$$

Z-TRANSFORM OF DELAYED SIGNAL

- What happens to $X(z)$ if we delay $x[n]$?

- Consider the signal:

$$x[n] = 3\delta[n-1] \leftrightarrow X(z) = 3z^{-1}$$

- Same signal, delayed by one:

$$w[n] = x[n-1] = 3\delta[n-2]$$

$$\leftrightarrow W(z) = 3z^{-2} = z^{-1}X(z)$$

Z-Transform Property: DELAY PROPERTY

A delay of one sample multiplies the z -transform by z^{-1} .

$$x[n - 1] \iff z^{-1} X(z)$$

Time delay of n_0 samples multiplies the z -transform by z^{-n_0}

$$x[n - n_0] \iff z^{-n_0} X(z)$$

DELAY EXAMPLE

- UNIT DELAY: find $y[n]$ via polynomials
 - $x[n] = \{3, 1, 4, 1, 5, 9, 0, 0, 0, \dots\}$

$$Y(z) = z^{-1} X(z)$$

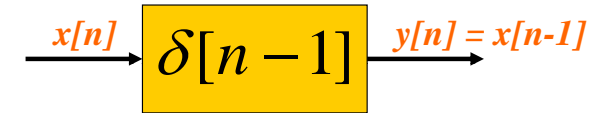
$$Y(z) = z^{-1} (3 + z^{-1} + 4z^{-2} + z^{-3} + 5z^{-4} + 9z^{-5})$$

$$Y(z) = 0z^0 + 3z^{-1} + z^{-2} + 4z^{-3} + z^{-4} + 5z^{-5} + 9z^{-6}$$

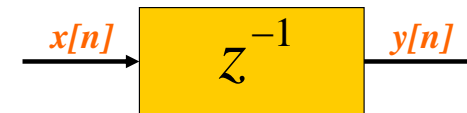
n	$n < 0$	0	1	2	3	4	5	6	$n > 6$
$y[n]$	0	0	3	1	4	1	5	9	0

EX. DELAY SYSTEM

- UNIT DELAY: find $h[n]$ and $H(z)$



$$H(z) = \sum \delta[n - 1] z^{-n} = z^{-1}$$



Z-Transform of FIR Filter

- CALLED the **SYSTEM FUNCTION**
 - because $h[n]$ is same as $\{b_k\}$

SYSTEM FUNCTION

$$H(z) = \sum_{k=0}^M b_k z^{-k} = \sum_{k=0}^M h[k] z^{-k}$$

$$y[n] = \sum_{k=0}^M b_k x[n - k] = \sum_{k=0}^M h[k] x[n - k]$$

FIR DIFFERENCE EQUATION

CONVOLUTION

Z-Transform of FIR Filter

- Get $H(z)$ DIRECTLY from the $\{b_k\}$
- Example 7.3 in the book:

$$y[n] = 6x[n] - 5x[n-1] + x[n-2]$$

$$\{b_k\} = \{6, -5, 1\}$$

$$H(z) = \sum b_k z^{-k} = 6 - 5z^{-1} + z^{-2}$$

GENERAL I/O PROBLEM

- Input is $x[n]$, find $y[n]$ (for FIR, $h[n]$)
- How to combine $X(z)$ and $H(z)$?

Example 7.5

$$x[n] = \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4]$$

$$\text{and } h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

$$\text{and } H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

FIR Filter = CONVOLUTION

$x[n], X(z)$	0	+1	-1	+1	-1			
$h[n], H(z)$	1	2	3	4				

	0	+1	-1	+1	-1			
		0	+2	-2	+2	-2		
			0	+3	-3	+3	-3	
				0	+4	-4	+4	-4

$y[n], Y(z)$	0	+1	+1	+2	+2	-3	+1	-4

$$y[n] = \sum_{k=0}^M b_k x[n-k] = \sum_{k=0}^M h[k] x[n-k]$$

CONVOLUTION

CONVOLUTION PROPERTY

- PROOF:

$$y[n] = x[n] * h[n] = \sum_{k=0}^M h[k] x[n-k]$$

$$Y(z) = \sum_{k=0}^M h[k] (z^{-k} X(z))$$

$$= \left(\sum_{k=0}^M h[k] z^{-k} \right) X(z) = H(z) X(z).$$

**MULTIPLY
z-TRANSFORMS**

CONVOLUTION EXAMPLE

- **MULTIPLY** the z-TRANSFORMS:

Example 7.5

$$x[n] = \delta[n - 1] - \delta[n - 2] + \delta[n - 3] - \delta[n - 4]$$

$$\text{and } h[n] = \delta[n] + 2\delta[n - 1] + 3\delta[n - 2] + 4\delta[n - 3]$$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

$$\text{and } H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

MULTIPLY $H(z)X(z)$

CONVOLUTION EXAMPLE

- Finite-Length input $x[n]$
- FIR Filter ($L=4$)

**MULTIPLY
Z-TRANSFORMS**

$$Y(z) = H(z)X(z)$$

$$= (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})(z^{-1} - z^{-2} + z^{-3} - z^{-4})$$

$$= z^{-1} + (-1 + 2)z^{-2} + (1 - 2 + 3)z^{-3} + (-1 + 2 - 3 + 4)z^{-4}$$

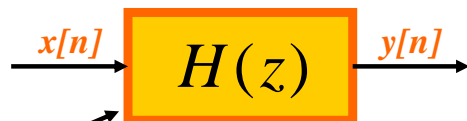
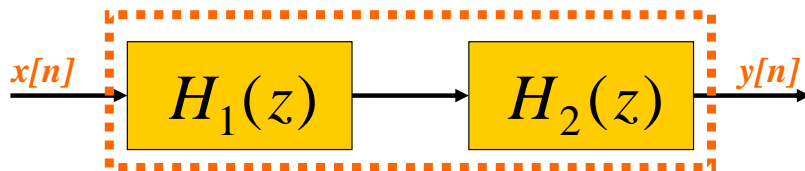
$$+ (-2 + 3 - 4)z^{-5} + (-3 + 4)z^{-6} + (-4)z^{-7}$$

$$= z^{-1} + z^{-2} + 2z^{-3} + 2z^{-4} - 3z^{-5} + z^{-6} - 4z^{-7}$$

$y[n] = ?$

Z-Transform Property: CASCADE EQUIVALENT

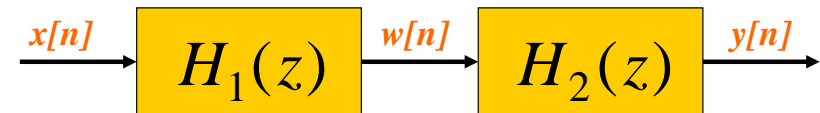
- Multiply the System Functions



**EQUIVALENT
SYSTEM**

$$H(z) = H_1(z)H_2(z)$$

CASCADE EXAMPLE

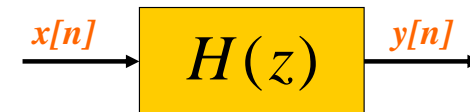


$$w[n] = x[n] - x[n - 1]$$

$$H_1(z) = 1 - z^{-1}$$

$$y[n] = w[n] + w[n - 1]$$

$$H_2(z) = 1 + z^{-1}$$



$$H(z) = (1 - z^{-1})(1 + z^{-1}) = 1 - z^{-2}$$

$$y[n] = x[n] - x[n - 2]$$