DSP First, 2/e

Lecture 21

Zeros of H(z) and the Frequency Domain

READING ASSIGNMENTS

- This Lecture:
  - Chapter 9, Sects. 9-5 & 9-6
- Other Reading:
  - Examples: Chapter 9, Sects. 9-7 & 9-8
    - ZEROS (and POLES)
    - Practical Bandpass Filters

LECTURE OBJECTIVES

- ZEROS and POLES
- Relate H(z) to FREQUENCY RESPONSE

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- *THREE DOMAINS:*
  - *Show Relationship for FIR:*

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

DESIGN PROBLEM

- Example:
  - Design a Lowpass FIR filter (Find b_k)
  - Reject completely 0.7π, 0.8π, and 0.9π
    - This is NULLING
  - Estimate the filter length needed to accomplish this task. How many b_k ?
- Z POLYNOMIALS provide the TOOLS

CONVOLUTION PROPERTY

- Convolution in the n-domain
  - SAME AS
- Multiplication in the z-domain

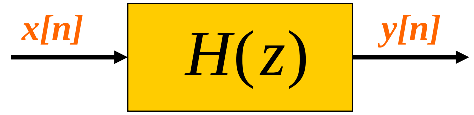
$$y[n] = h[n] * x[n] \quad \leftrightarrow \quad Y(z) = H(z)X(z)$$

$$y[n] = x[n] * h[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

FIR Filter

MULTIPLY z-TRANSFORMS

CONVOLUTION EXAMPLE



$$x[n] = \delta[n-1] + 2\delta[n-2] \qquad h[n] = \delta[n] - \delta[n-1]$$

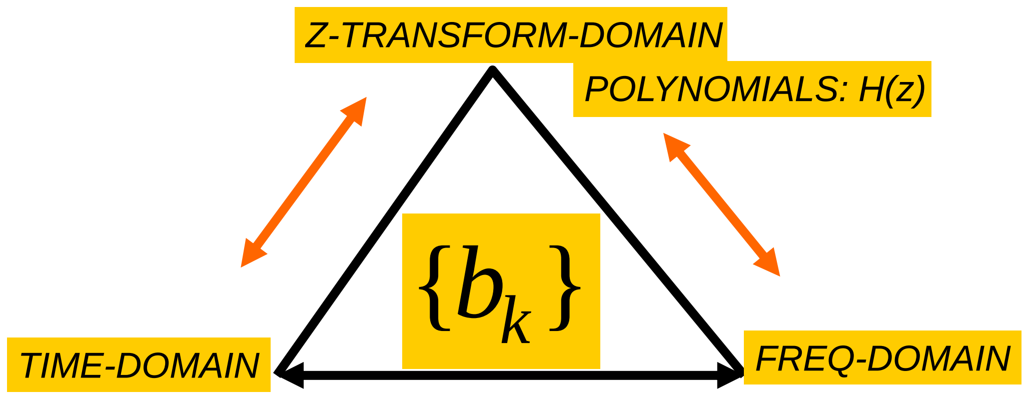
$$y[n] = x[n] * h[n]$$

$$X(z) = z^{-1} + 2z^{-2} \qquad H(z) = 1 - z^{-1}$$

$$Y(z) = (z^{-1} + 2z^{-2})(1 - z^{-1}) = z^{-1} + z^{-2} - 2z^{-3}$$

$$y[n] = \delta[n-1] + \delta[n-2] - 2\delta[n-3]$$

THREE DOMAINS



$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \qquad H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

FREQUENCY RESPONSE ?

- Same Form:

$$z = e^{j\hat{\omega}}$$

$\hat{\omega}$ – Domain

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k (e^{j\hat{\omega}})^{-k}$$

z – Domain

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

SAME COEFFICIENTS

H(z) as ANALYSIS TOOL

- H(z) is a COMPLEX-VALUED function of a COMPLEX VARIABLE z.
- Shape of H(z), and the Frequency Response, is dominated by zeros (H(z)=0) and poles (H(z)=∞)
- Can we use tools of POLYNOMIALS (e.g., roots and factoring) to make analysis easier?

ZEROS of H(z)

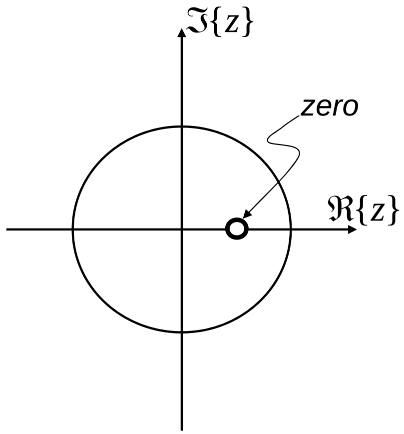
$$H(z) = 1 - \tfrac{1}{2} z^{-1}$$

- Find z, where H(z)=0

$$1 - \tfrac{1}{2} z^{-1} = 0\ ?$$

$$z - \tfrac{1}{2} = 0$$

$$\text{Zero at} : z = \tfrac{1}{2}$$



ZEROS of H(z) – example 2

- Find z, where H(z)=0
  - Interesting when z is ON the unit circle.

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

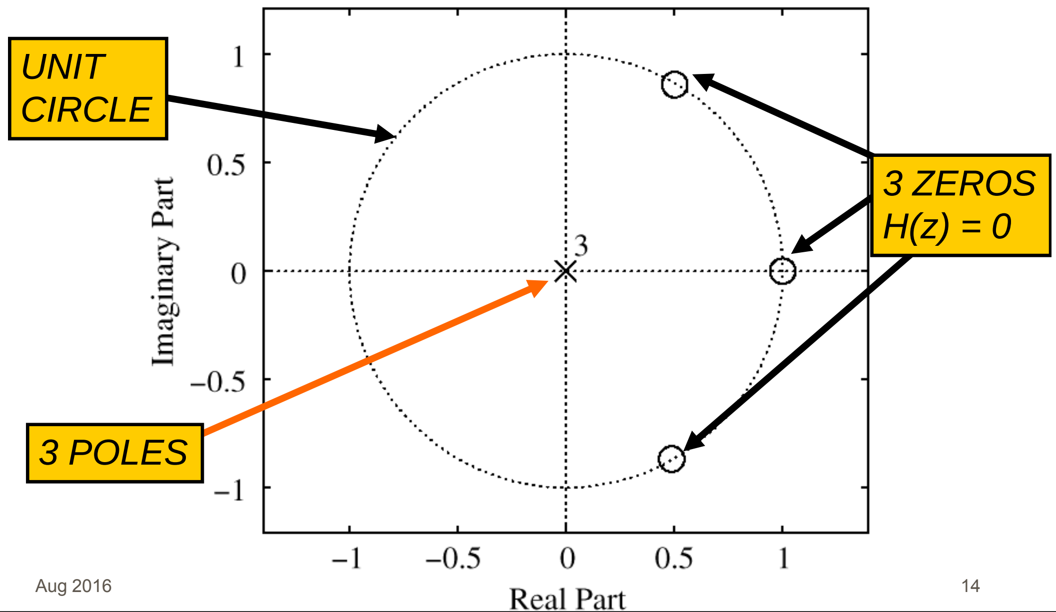
$$H(z) = (1 - z^{-1})(1 - z^{-1} + z^{-2})$$

$$\text{Roots} : z = 1, \tfrac{1}{2} \pm j\tfrac{\sqrt{3}}{2}$$

$$e^{\pm j\pi/3}$$

Recall: Roots occur in Conjugate pairs when coefficients are real

PLOT ZEROS in z-DOMAIN



POLES of H(z)

- Find z, where $H(z) \to \infty$
  - FIR only has poles at z=0

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(z) = \frac{z^3 - 2z^2 + 2z \;\; -1}{z^3}$$

$$\text{Three Poles at} : z = 0$$

FREQUENCY RESPONSE ?

- Same Form:

$$z = e^{j\hat{\omega}}$$

$\hat{\omega}$ – Domain

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k (e^{j\hat{\omega}})^{-k}$$

z – Domain

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

SAME COEFFICIENTS

FREQ RESPONSE from System Function

- Relate H(z) to FREQUENCY RESPONSE

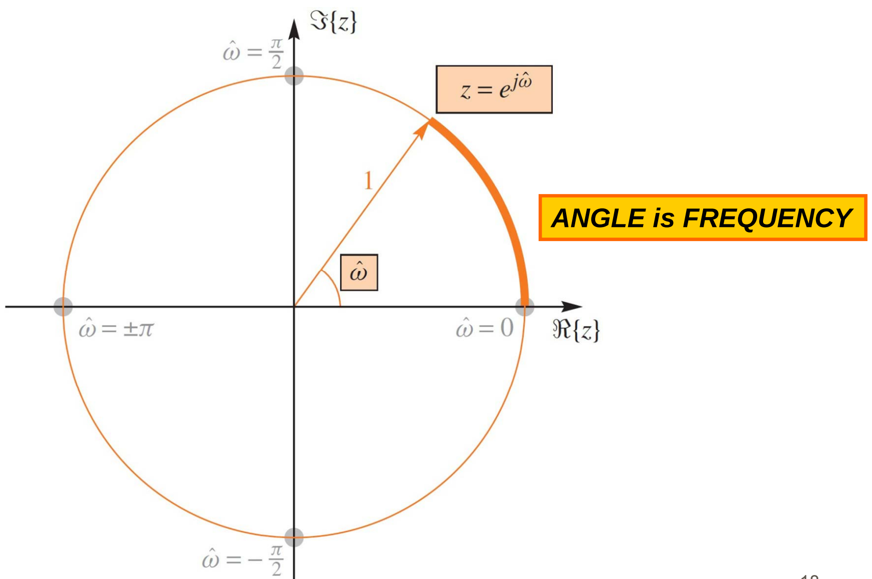
$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- EVALUATE H(z) on the **UNIT CIRCLE**
  - ANGLE is same as FREQUENCY

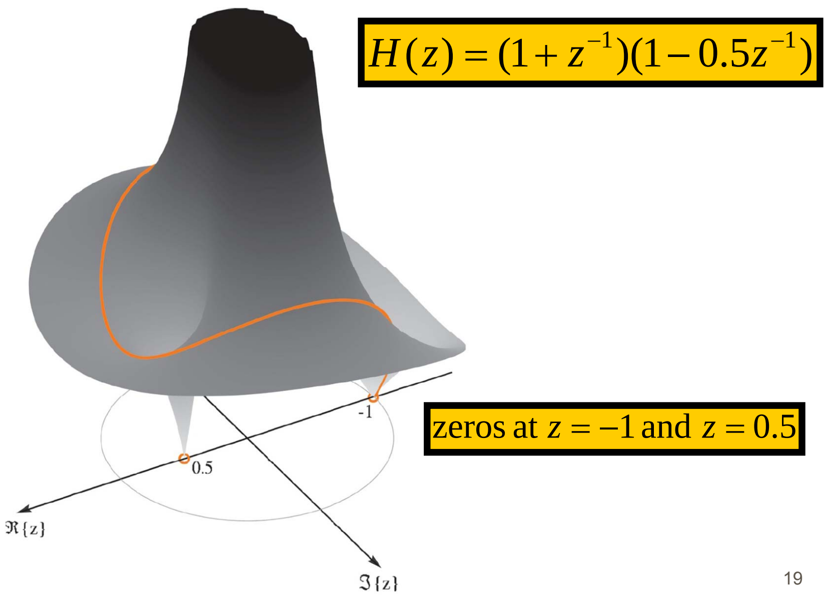
$$z = e^{j\hat{\omega}} \qquad \text{(as } \hat{\omega} \text{ varies)}$$

$$\text{defines a CIRCLE, radius} = 1$$

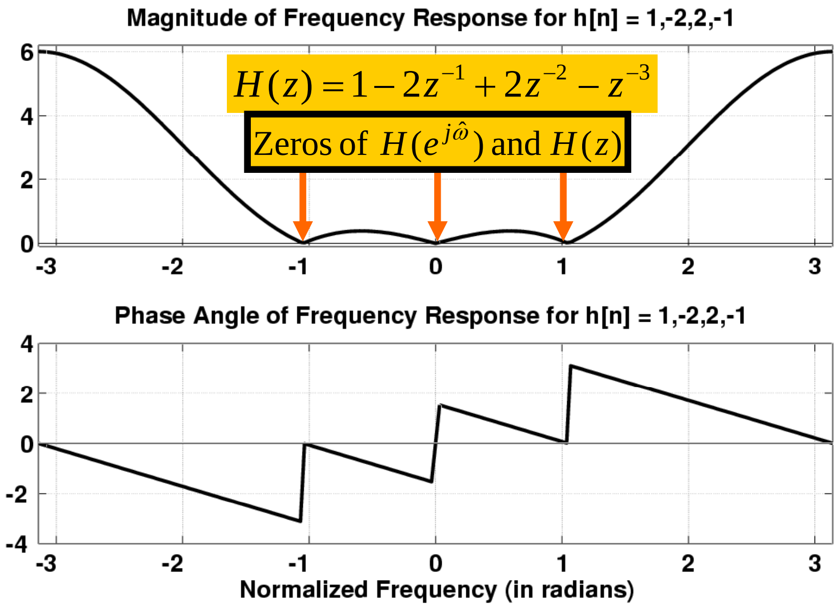
$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$



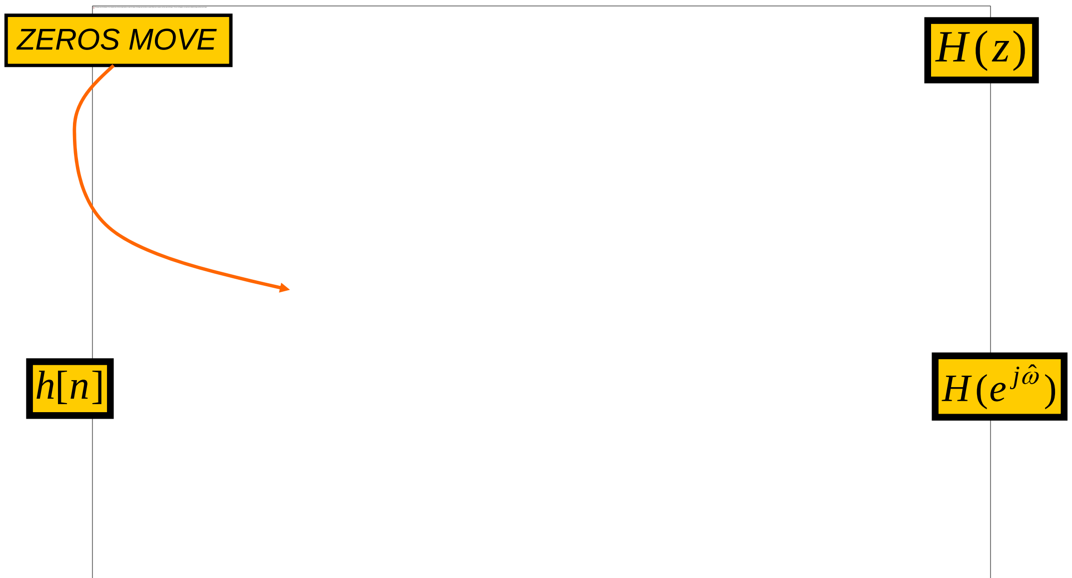
Evaluate H(z) on Unit Circle



FIR Frequency Response



3 DOMAINS MOVIE: FIR



4 MOVIES @ WEBSITE

- http://dspfirst.gatech.edu/chapters/07ztrans/demos/3_domain/index.html
- 3 DOMAINS MOVIES: FIR Filters
  - Two zeros moving around UC and inside
  - Three zeros; one held fixed at $z = -1$
  - Ten zeros; 9 equally spaced around UC; one moving
  - Ten zeros; 8 equally spaced around UC; two moving

NULLING PROPERTY of H(z)

- When H(z)=0 on the unit circle.
  - Find inputs x[n] that give zero output

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}$$

x[n] → $H(z)$ → y[n]

$$H(e^{j\pi/3}) = ?$$

$$x[n] = e^{j(\pi/3)n}$$

$$y[n] = H(e^{j(\pi/3)}) \cdot e^{j(\pi/3)n}$$

NULLING PROPERTY of H(z)

- Evaluate H(z) at the input "frequency"

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}$$

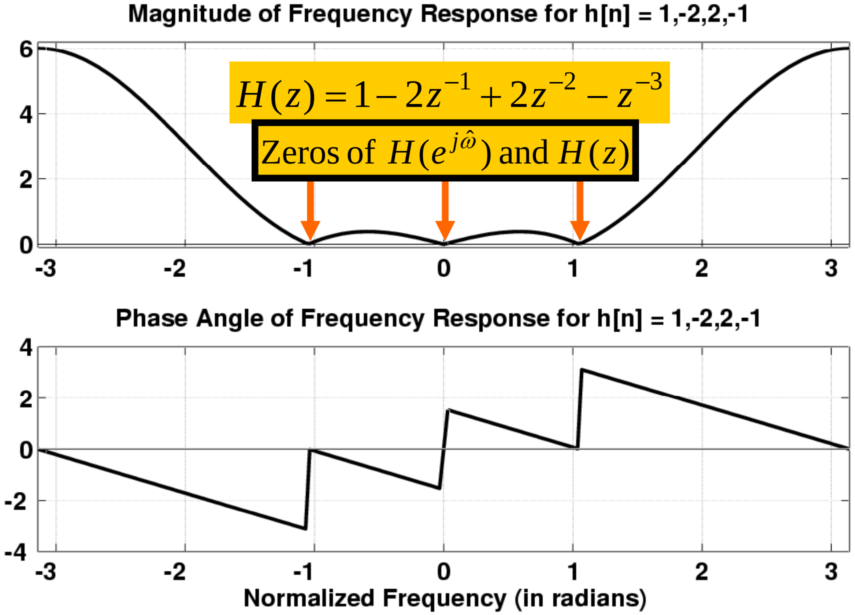
$$y[n] = H(e^{j\pi/3}) \cdot e^{j(\pi/3)n}$$

$$y[n] = (1 - 2e^{-j\pi/3} + 2e^{-j2\pi/3} - e^{-j3\pi/3}) \cdot e^{j(\pi/3)n}$$

$$(1 - 2(\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}) + 2(-\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}) - (-1))$$

$$y[n] = (1 - 1 + j\sqrt{3} - 1 - j\sqrt{3} + 1) \cdot e^{j(\pi/3)n} = 0$$

FIR Frequency Response



DESIGN PROBLEM

- Example:
  - Design a Lowpass FIR filter (Find b_k)
  - Reject completely 0.7π, 0.8π, and 0.9π
  - Estimate the filter length needed to accomplish this task. How many b_k ?
- Z POLYNOMIALS provide the TOOLS

NULLING FILTER DESIGN

- PLACE ZEROS to make y[n] = 0

Need 6 ZEROS where H(z) = 0

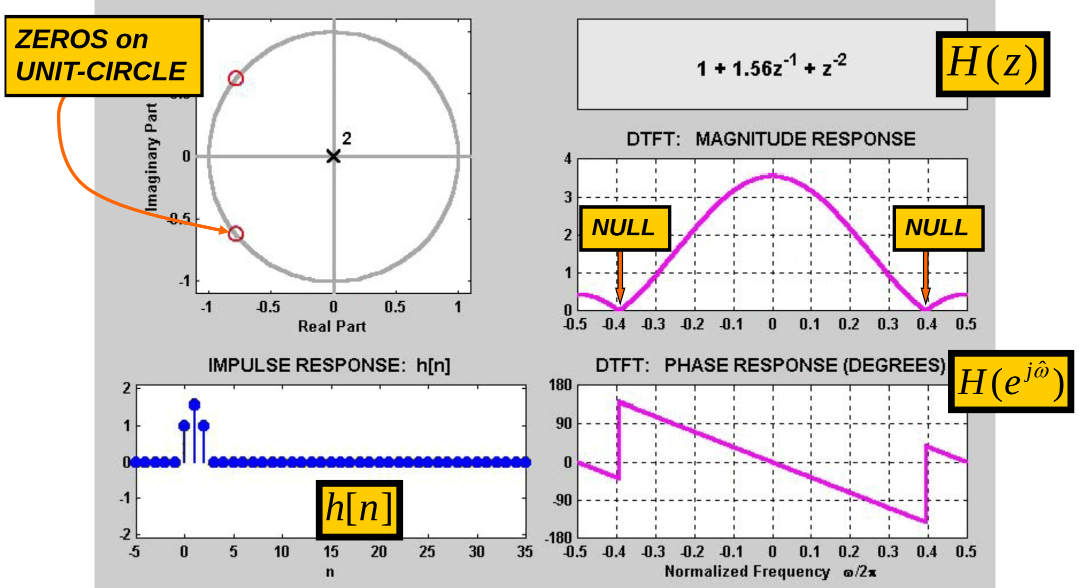
$$H(z_k) = 0, \quad \text{for } z_k = e^{\pm j0.7\pi}, e^{\pm j0.8\pi}, e^{\pm j0.9\pi}$$

$$x[n] = e^{j0.8\pi n} \quad \Rightarrow \quad y[n] = H(e^{j0.8\pi})e^{j0.8\pi n}$$

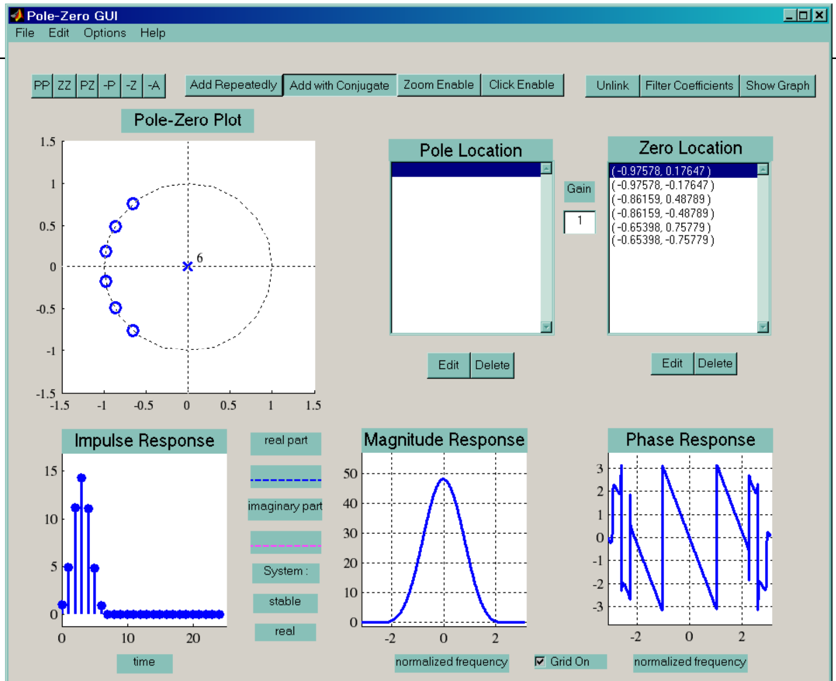
- 6^{th} order FIR has 7 filter coefficients

$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4} + b_5 z^{-5} + b_6 z^{-6}$$

3 DOMAINS MOVIE: FIR



PeZ Demo: Zero Placing



One zero, two zeros, ...

We usually want filters with real coefficients

$$H(z) = 1 - az^{-1} \quad \Rightarrow \quad H(z) = 0 \;\; @ \;\; z = a$$

If we want to block sinusoid with $\hat{\omega} = \pm 0.8\pi$

$$H(z_k) = 0 \quad \text{for } z_k = e^{\pm j0.8\pi}$$

$$\Rightarrow H(z) = z^{-2}(z - e^{j0.8\pi})(z - e^{-j0.8\pi})$$

$$= z^{-2}(z^2 - z(e^{j0.8\pi} + e^{-j0.8\pi}) + 1)$$

$$= 1 - 2(\cos 0.8\pi)z^{-1} + z^{-2} = 1 + 1.618z^{-1} + z^{-2}$$

z^{-2} needed for causality

$$h[0] = 1, \quad h[1] = 1.618, \quad h[2] = 1$$

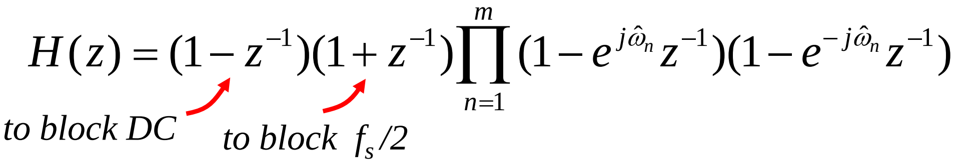
Block Multiple Frequencies

Want to totally block: $\hat{\omega}_1, \hat{\omega}_2, \cdots, \hat{\omega}_m$

H(z) *must have zeros at:* $z = e^{\pm j\hat{\omega}_1}, e^{\pm j\hat{\omega}_2}, \cdots, e^{\pm j\hat{\omega}_m}$

To block $\hat{\omega} = 0$ or π *must have zero at* z *=1 or -1*

So, the general form becomes:



On the other hand: Not much control over other frequencies

L-pt RUNNING SUM H(z)

$$H(z) = \sum_{k=0}^{L-1} z^{-k} = \frac{1 - z^{-L}}{1 - z^{-1}} = \frac{z^L - 1}{z^{L-1}(z-1)}$$

$$z^L - 1 = 0 \quad \Rightarrow \quad z^L = 1 = e^{j2\pi k}$$

$$z = e^{j(2\pi/L)k} \quad \text{for } k = 1, 2, \ldots L-1$$

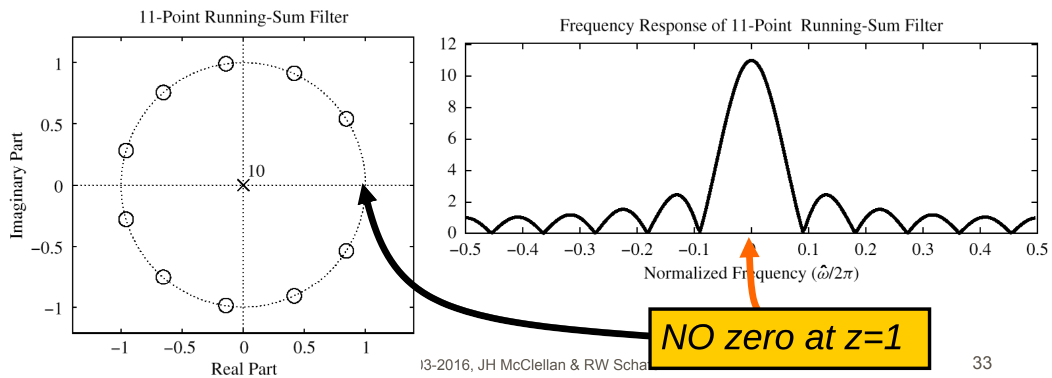
ZEROS on UNIT CIRCLE

(z-1) in denominator cancels k=0 term

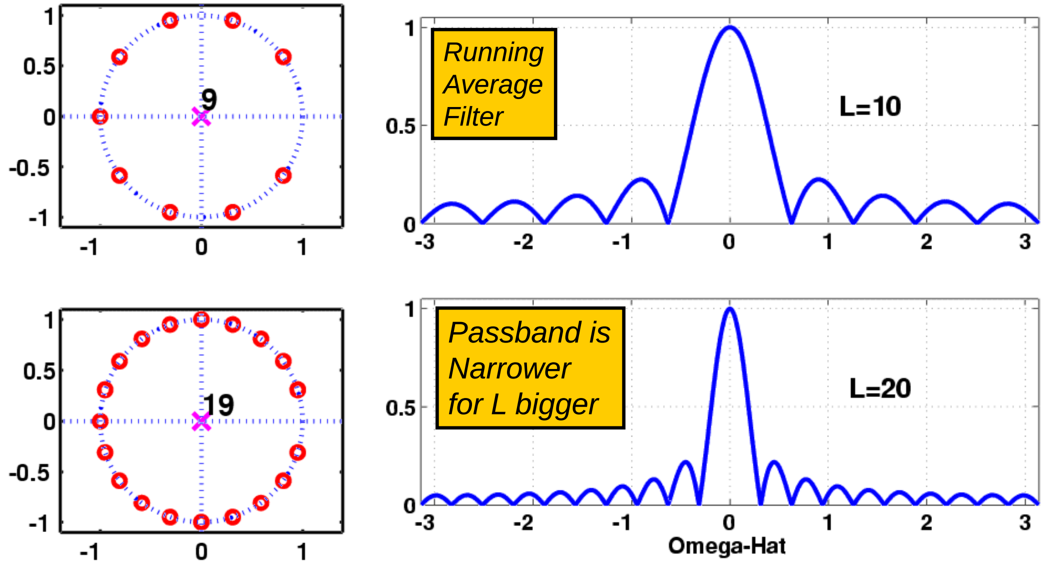
11-pt RUNNING SUM H(z)

$$H(z) = \sum_{k=0}^{10} z^{-k}$$

$$H(z) = (1 - e^{j2\pi/11} z^{-1})(1 - e^{j4\pi/11} z^{-1}) \cdots (1 - e^{j20\pi/11} z^{-1})$$



FILTER DESIGN: CHANGE L



3 DOMAINS MOVIE: FIR BPF